

# COMPUTATION BOOK

NAME

Number

Course.....

Used from ..... 19....., to ..... 19.....

HARVARD COOPERATIVE SOCIETY

1400 Mass. Ave., Cambridge, Mass.

40 Mass. Ave., Cambridge, Mass.

# I. Venus Surface Temperatures

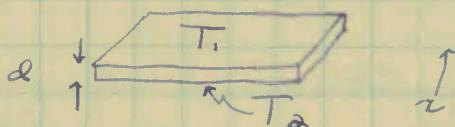
To treat this subject thoroughly we first need to develop three topics from first principles:

- Theory of Heat Conduction in Semi-Infinite Solids.
- Electromagnetic and thermal skin depth, and
- Faraday equations for dielectrics.

## A. Theory of Heat Conduction in a Semi-Infinite Solid

Reference: Carslaw + Jaeger, 2nd ed., chaps. 1+2.

In the ff. the thermal conductivity  $K$  is defined by this exp't:



Consider a surface of area  $S$  + thickness  $d$ , the top face of which is maintained at a temperature  $T_1$ , the bottom face of which is maintained at a temperature  $T_2$ .  $T_2 > T_1$ . At equilibrium, it is found that the quantity of heat,  $Q$ , which flows through  $S$  in time  $\Delta t$  is

$$Q = \frac{K(T_2 - T_1) S \Delta t}{d} \quad (1)$$

The greater  $ST$ , the greater  $Q$ . The greater  $d$ , the smaller  $Q$ . The units of  $K$  are then

$$\begin{aligned} & \text{cal cm}^{-2} \text{ sec}^{-1} (K^\circ)^{-1} \text{ cm} \\ & = \text{cal cm}^{-1} \text{ sec}^{-1} (K^\circ)^{-1} \end{aligned}$$

In most applications,  $K = K(r, \theta, T)$  is assumed. Actually, a better approx is

$$K = K_0(1 - \beta T) \quad (2)$$

where  $-\beta$  is small, and, for most substances, negative.

From (1) we see that the flux of heat in the  $x$ -direction is

$$F_x = -K \frac{\partial T}{\partial x}$$

The minus sign exists because the heat flows from the hotter to the cooler surface.

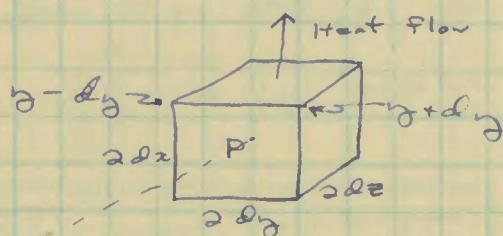


In three dimensions,

$$\vec{F} = -K \vec{\nabla} T. \quad (3)$$

$$= -K \left( \frac{\partial F_x}{\partial x} \hat{i} + \frac{\partial F_y}{\partial y} \hat{j} + \frac{\partial F_z}{\partial z} \hat{k} \right)$$

Now consider a rectangular parallelepiped of infinitesimal dimensions:



$F_x$  is flux across plane at P.

P is at center of parallelepiped.  $\therefore$  heat flows in at a rate  $(F_x - \frac{\partial F_x}{\partial x} dx) 2dy 2dz$ , and out at a rate  $(F_x + \frac{\partial F_x}{\partial x} dx) 2dy 2dz$ . Thus the net rate of gain of heat by conduction is

$$4F_x dy dz - 4 \frac{\partial F_x}{\partial x} dx dy dz - 4F_x dy dz + 4 \frac{\partial F_x}{\partial x} dx dy dz$$

$$= -8 dx dy dz \frac{\partial F_x}{\partial x}.$$

Similarly, for heat flow across the other two sets of 11 faces. Adding we have net rate of gain of heat

$$= -8 \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx dy dz \\ = -8 \vec{\nabla} \cdot \vec{F} d^3x.$$

Now this quantity is also  $8 \frac{\partial H}{\partial x} d^3x$ , where  $H$  is the heat content per unit volume. ~~But~~ The factor 8 follows simply from the geometry; if the volume of the parallelepiped had been defined as  $d^3x$ , the factor would be unity. Now  $\Delta H = \rho c \Delta T$ , where  $c$  is the specific heat per unit mass. This is the definition of the specific heat.

$$\therefore -8 \vec{\nabla} \cdot \vec{F} d^3x = 8 \rho c \frac{\partial T}{\partial x} d^3x$$

$$\therefore \rho c \frac{\partial T}{\partial x} + \vec{\nabla} \cdot \vec{F} = 0$$

For a homogeneous, isotropic solid with  $K = K(T)$ , we know from Eq. 31 that  $\vec{F} = -K \vec{\nabla} T$ .

$$\therefore \rho c \frac{\partial T}{\partial x} - K \nabla^2 T = 0, \quad \text{or}$$

$$\boxed{\frac{\partial T}{\partial x} = \kappa \nabla^2 T} \quad (4)$$

where  $\kappa = K/\rho c$  is the thermal diffusivity. When we have a steady state,  $\partial T/\partial x = 0$ , and the eq. of heat conduction, (4), reduces to Laplace's equation  $\nabla^2 T = 0$ .

For solids the method of heating has little effect on the value of the specific heat.  
 $\therefore C \approx C_p$ .

Dimensions of the diffusivity,  $\kappa$ :

$$\frac{\text{cal cm}^{-2} \text{ sec}^{-1} (\text{K})^{-1}}{\text{gm cm}^{-3} \text{ cal gm}^{-1} (\text{K})^{-1}}$$

$$= \text{cm}^2 \text{ sec}^{-1}$$

To solve the diff. eq. (4), we must specify initial and boundary condns.

Consider a planar surface subject to periodic insulation. We expect the surface temperature to be a sinusoidal function of time. I.e., for plane-parallel geometry and a semi-infinite solid, the differential eq.

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \quad (5)$$

has the anticipated solution

$$T = u(z) e^{i(wt - \phi)} \quad (6)$$

The desired boundary conditions are that  $T \rightarrow T'$  as  $z \rightarrow \infty$ , and  $T \rightarrow T_0 \cos(wt - \phi)$  as  $z \rightarrow 0$ .

The solution (6) has period  $P = 2\pi/w$ .

Substituting (6) into (5), we find

$$\frac{\partial T}{\partial t} = u(z) i w e^{i(wt - \phi)}$$

$$\kappa \frac{\partial^2 T}{\partial z^2} = \kappa \frac{\partial^2 u}{\partial z^2} e^{i(wt - \phi)}$$

$$\therefore i w u(z) = \kappa \frac{\partial^2 u}{\partial z^2}$$

$$\therefore \frac{\partial^2 u}{\partial z^2} = \frac{i w}{\kappa} u$$

We recognize that this auxiliary total differential eq.  $\frac{\partial^2 u}{\partial z^2} = \frac{i\omega}{\kappa} u$  is also periodic.

$$\therefore \text{Let } u = A e^{i\beta(z - \pi)}$$

$$\therefore \frac{\partial u}{\partial z} = i\beta u$$

$$\frac{\partial^2 u}{\partial z^2} = i\beta \frac{\partial u}{\partial z} = i^2 \beta^2 u = -\beta^2 u$$

$$\therefore -\beta^2 u = \frac{i\omega}{\kappa} u$$

$$\beta^2 = -\frac{i\omega}{\kappa}$$

$$\text{Now note that } \left[ \frac{1+i}{\sqrt{2}} \right]^\alpha = \frac{1+2i-1}{\alpha} = i.$$

$$\therefore \sqrt{i} = \pm \frac{1+i}{\sqrt{2}}$$

$$\sqrt{-i} = \pm \sqrt{-1} \sqrt{i} = \pm i \frac{1+i}{\sqrt{2}} = \pm \frac{i-1}{\sqrt{2}}.$$

$$\therefore \beta = \sqrt{-i} \sqrt{\frac{\omega}{\kappa}} = \pm \frac{i-1}{\sqrt{2}} \sqrt{\frac{\omega}{\kappa}} = \pm (i-1) \sqrt{\frac{\omega}{2\kappa}}$$

$$\therefore u = A e^{-(1+i)\sqrt{\frac{\omega}{2\kappa}}(z - \pi)}$$

The + sign in the exponent causes a divergence for  $z \rightarrow \infty$ .

$$\therefore u = A e^{-(1+i)\sqrt{\frac{\omega}{2\kappa}}(z - \pi)}$$

$$\begin{aligned} \therefore T &= A e^{-\sqrt{\frac{\omega}{2\kappa}}(z - \pi)} e^{-i\sqrt{\frac{\omega}{2\kappa}}(z - \pi)} e^{i(\omega t - \varphi)} \\ &= A e^{-\kappa(z - \pi)} \cos \left\{ \omega t - \varphi - \kappa(z - \pi) \right\} \end{aligned}$$

where  $\kappa = \sqrt{\frac{\omega}{2\kappa}}$ . We are at liberty to set  $\pi = 0$ , but not  $\varphi = 0$ , because of the oscillator. Either cos or sin may be chosen by suitable adjustment of  $\varphi$ . We choose cos, and we have our final solution.

The more rapid attenuation of the higher frequency components in a Fourier decomposition of the temporal oscillation has the following consequence: A complicated insulation pattern at the surface takes a simpler and more sinusoidal form as we probe greater depths. Note that Mayer's <sup>3</sup> data can be represented sinusoidally while Drakos' <sup>10</sup> data cannot, in apparent contradiction. However, Drakos' data are not good enough to permit a <sup>unique</sup> specification of the functional form of  $T_0(x)$ .

$$T = T_0 e^{-kz} \cos(\omega t - kz - \varphi) \quad (7)$$

Note that the diff. eq. of heat conduction permits an additive const., so the final sol. is

$$T = T' + T_0 e^{-kz} \cos(\omega t - kz - \varphi) \quad (8)$$

This solution exhibits a temperature wave of wave number  $k$ , and therefore of wavelength

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{\frac{\omega}{2\kappa}}} = \sqrt{\frac{8\pi^2\kappa}{\omega}} = \sqrt{\frac{4\pi\kappa}{\nu}} \quad (9)$$

where  $\nu$  is the frequency  $\omega/2\pi$ . Knowing the frequency and the wavelength, we readily find the velocity of propagation of the thermal wave

$$v = v\lambda = \frac{\omega}{2\pi} \cdot \frac{2\pi}{\sqrt{\frac{\omega}{2\kappa}}} = \sqrt{2\kappa\omega}. \quad (10)$$

The amplitude of the temperature oscillation diminishes with depth as

$$e^{-kz} = e^{-\sqrt{\frac{\omega}{2\kappa}} z} = e^{-2\pi z/\lambda}.$$

∴ the higher frequency components of a Fourier decomposition of the solution (7) are attenuated more rapidly.

There is a progressive phase lag

$$kz = \sqrt{\frac{\omega}{2\kappa}} z \quad (11)$$

which increases with  $\omega$ .

Superficial intercomparison of 3- and 10 cm results — e.g., Sauer + Kellogg, p. 248 — suggests shear angle  $\tan \theta_s = 30^\circ = \frac{30}{180} \pi = 0.53$  rad. The shear lag diff. if  $z=1$  should be  $K\Delta z = \tau k$ .  
 $\therefore k = \frac{0.53}{7} \approx 0.075$  cm for dry quartz sand.  
 $k = 2\pi/\lambda = \frac{6.28}{5.1 \times 10^2} = 0.012$  cm<sup>-1</sup> for  $P_{sid}^2 \approx 118^2$ .

For agreement,  $k$  must be 6 times larger,  $\therefore \lambda$  36 times smaller, suggesting much drier stuff than medium fine dry quartz sand. With such a revised  $k$ ,  $\lambda \approx 85$  cm.

But what the observed  $\theta_s$  shear lag is is a problem because of Drucker's  $T_B = 622 + 39 \cos(\varphi \pm 17^\circ)^\circ K$ . The I, with Mayr et al.'s  $T_B = 621 + 73 \cos(\varphi - 11.7^\circ)^\circ K$  gives shear lags either of  $29^\circ$  or of  $5^\circ$ , depending on whether we are before or after inf. conj. In the other case, w.  $Kz = 5^\circ$ ,  $k = 0.012$  cm<sup>-1</sup> in perfect agreement with dry quartz sand.

If  $w$  were 10x greater,  $k$  would also have to be 10x greater, corresponding, e.g., to sandy clay w. 15% moisture content. (But note also that  $\dot{\gamma}_{rock} \approx 2 \times 10^{-3}$  cm<sup>2</sup> sec<sup>-1</sup>, so by no means a clear indication of granular material.) ... possibly rapid rotation  $\Rightarrow$  wet surface; slow rotation  $\Rightarrow$  dry surface?

Assume that the surface of  $\oplus$  has the same thermal properties as medium fine dry quartz sand.

$$\therefore \kappa \approx 2 \times 10^{-3} \text{ cm}^2 \text{ sec}^{-1}.$$

Now on the surface of  $\oplus$ , the <sup>solar</sup> sidereal period of the  $\odot$  is given by

$$\frac{1}{P_{\text{sol}}} = \frac{1}{P_{\text{rev}}} \Theta \frac{1}{P_{\text{rot}}} \quad (12)$$

With  $P_{\text{rev}} = 225^\circ$  and  $P_{\text{rot}} = \Theta 250^\circ$  (retrograde rot.)

$$P_{\text{sol}}^{-1} = \frac{225 + 250}{225 \cdot 250} = \frac{475}{225 \cdot 250}$$

$$\therefore P_{\text{sol}} \approx 118^\circ \quad (13)$$

$\therefore$  the sun rises and sets slightly less than twice a year.

$$\therefore \omega = \frac{2\pi}{P} = \frac{2\pi}{118 \times 8.64 \times 10^4} = 6.15 \times 10^{-7} \text{ rad/sec.}$$

$$\therefore \lambda = \sqrt{\frac{8\pi^2 \kappa}{\omega}} = \left[ \frac{8 \times \pi^2 \times 2 \times 10^{-3}}{6.15 \times 10^{-7}} \right]^{1/2} = [2.58 \times 10^{-5}]^{1/2}$$

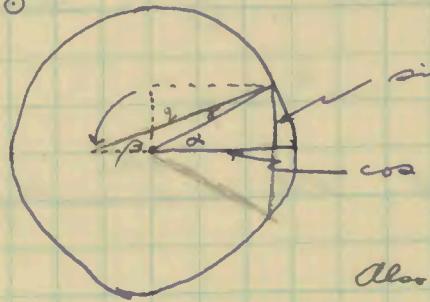
$$= 5.1 \times 10^2 \text{ cm} = 5.1 \text{ meters.}$$

$$v = \frac{\omega}{2\pi} \lambda = \frac{6.15 \times 10^{-7} \times 5.1 \times 10^2}{2\pi} = 5 \times 10^{-5} \text{ cm/sec,}$$

the propagation velocity of the thermal wave.

At  $z = \lambda$ , the phase lag is  $k\lambda = 2\pi$  radians, or back in phase again. E.g., at  $z = 5 \text{ cm}$ , the phase lag  $kz$   $= \frac{2\pi}{\lambda} z = \frac{2\pi}{500} \times 5 = 0.063$  radians. At  $10 \text{ cm}$ ,  $0.13$  radians. At  $40 \text{ cm}$ ,  $0.52$  radians.  $\therefore$  if emission at wavelength  $\lambda$  comes from depth  $z \approx \lambda$ , a small phase lag should be observed after intercomparing observations. The phase lag itself determines  $k$ .

Unit 0



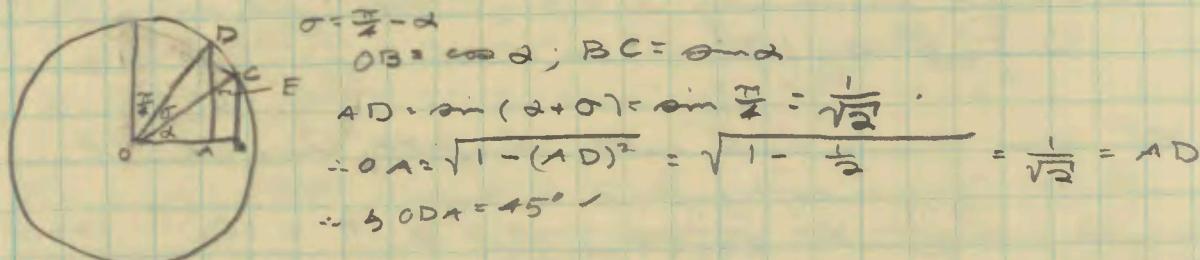
$$\begin{aligned}r^2 &= \sin^2 \alpha + (\cos \alpha)^2 \\&= \sin^2 \alpha + \sin^2 \alpha + 2 \sin \alpha \cos \alpha + \cos^2 \alpha \\r &= \sqrt{\sin^2 \alpha + 2 \sin \alpha \cos \alpha + \cos^2 \alpha}\end{aligned}$$

$$\text{Also } r^2 = 1^2 + \sin^2 \alpha - 2 \sin \alpha \cos (\pi - \beta - \alpha)$$

$$\begin{aligned}&= 2 \sin \alpha \cos \alpha - 2 \sin \alpha \cos (\pi - \beta - \alpha) \\&\therefore \cos \alpha = -\cos (\pi - \beta - \alpha)\end{aligned}$$

$$\begin{aligned}\pi - \frac{\pi}{2} - \alpha + \gamma &= \pi - \frac{\pi}{2} - \beta \quad \gamma = \alpha - \beta \\&\therefore \cos \alpha = -\cos (\pi - \beta - \alpha + \beta) = \cos \alpha\end{aligned}$$

$$\text{Now } (\cos \alpha + \sin \alpha)^2 = \sin^2 \alpha + r^2 = 1 - 2 \sin^2 \alpha - 2 \sin \alpha \cos (\pi - \alpha)$$



$$\text{In } \triangle DOE, \angle ODE = 45^\circ. \sigma = 45^\circ - \alpha. \therefore \angle OED = \pi - \frac{\pi}{2} - \frac{\pi}{2} - \alpha = \frac{\pi}{2} + \alpha$$

$$K_k = g c \sqrt{\frac{\omega}{2\pi}} = g c \sqrt{\omega} \sqrt{\frac{\omega}{2}}.$$

$$\therefore F_o = g c \sqrt{\omega} \sqrt{\omega} T.$$

$$\text{But } F_o = \sigma T_o^4 \text{ at max temp.} \therefore \sigma T_o^4 = g c \sqrt{\omega} \sqrt{\omega} T_o, \text{ and}$$

$$T_o^3 = \frac{g c \sqrt{\omega}}{\sigma} \sqrt{\omega}. \text{ Taking values from ff. pg.}$$

$$T_o^3 = \frac{1.4 \times 10^{-2} \times 4.0 \times 10^7}{5.7 \times 10^{-5}} \times 7.9 \times 10^{-4} = 8.1 \times 10^6$$

$\therefore T_o = 200^\circ K$ ,  $\Delta T = 400^\circ K$   $\neq$  diurnal temp.  
 variation; just about the value derived on more sophisticated grounds. For  $\sigma + \theta$ ,  $\omega \rightarrow T_o$  by  $200^\circ \approx 1.8$ .  
 This contradicts obs. Why?

The heat flux at the surface is

$$\begin{aligned}
 F_z &= -K \left( \frac{\partial T}{\partial z} \right)_{z=0} \\
 &= -K \left\{ -k T_0 e^{-kz} \cos(\omega t - kz - \varphi) \right. \\
 &\quad \left. + T_0 e^{-kz} k \sin(\omega t - kz - \varphi) \right\}_{z=0} \\
 &= +K k T_0 \left\{ \cos(\omega t - \varphi) \pm \sin(\omega t - \varphi) \right\} \quad (14)
 \end{aligned}$$

From Dwight 401.2, p. 74,

$p \cos A + q \sin A = r \cos(A - \Phi)$ , a Fourier composition,  
where  $r = \sqrt{p^2 + q^2}$ , and  $\Phi = \arccos \frac{p}{r} = \arcsin \frac{q}{r}$ .

Here,  $p=1$ ,  $q=-1$ .  $\therefore r = \sqrt{2}$  and  $\Phi = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$ .

$$\therefore F_z = \sqrt{2} K k T_0 \cos(\omega t - \varphi + \frac{\pi}{4}) \quad (15)$$

At  $z=0$ , we have, by definition of the insulation,

$$F = F_0 \cos(\omega t - \varphi'). \quad (16)$$

$\therefore$  from (15),

$$F_0 = \sqrt{2} K k T_0$$

Relation between insulation at surface,  
surface thermal properties, & maximum value of  
surface time variable part of the  
surface temperature.

and

$$\varphi' = \varphi - \frac{\pi}{4}$$

$\therefore$  from (17),

$$T = \frac{F_0 e^{-kz}}{\sqrt{2} K k} \cos(\omega t - kz - \varphi' - \frac{\pi}{4}) \quad (17).$$

where  $\varphi'$  is the phase of the insulation.

Work in the other way, to get  $z(\lambda)$ :

$$3\text{ cm: } e^{-kz_3} = 73^\circ; \quad 10\text{ cm: } e^{-kz_{10}} = 39^\circ.$$

$$\therefore e^{-k(z_3 - z_{10})} = 73/39 = 1.88.$$

$$\therefore k(z_{10} - z_3) = 0.63.$$

$$k = \sqrt{\frac{\omega}{2\pi c}} = \frac{7.9 \times 10^{-4}}{\sqrt{2 \times 2 \times 10^{-3}}} = \frac{7.9 \times 10^{-4}}{6.3 \times 10^{-2}} = 1.25 \times 10^{-2}.$$

$$\therefore \Delta z = \frac{0.63}{1.25 \times 10^{-2}} = 50 \text{ cm.}$$

$$\frac{F_o e^{-kz_3}}{g c \sqrt{\epsilon} \sqrt{\omega}} = 73$$

$$\frac{F_o e^{-kz_{10}}}{g c \sqrt{\epsilon} \sqrt{\omega}} = 39 \quad \text{2 degs. in 3 unknowns}$$

Assume  $e^{-kz_{10}} \ll e^{-kz_3}$ .

$$\therefore \frac{F_o}{g c \sqrt{\epsilon} \sqrt{\omega}} e^{-kz_3} \approx \sqrt{3 \times 39 + 12}$$

$$73 - 39 = 34 \approx \frac{F_o}{g c \sqrt{\epsilon} \sqrt{\omega}} [1 - k z_3 - 1 + k z_{10}] = \frac{F_o}{g c \sqrt{\epsilon} \sqrt{\omega}} k \Delta z$$

$$\therefore F_o = \frac{34 \times 1.25 \times 10^{-2} \times 4.2 \times 10^7 \times 7.9 \times 10^{-4}}{1.25 \times 10^{-2} \times 50} = 2.5 \times 10^4 \text{ erg cm}^{-2} \text{ sec}^{-1} !$$

Upward 10% for emissivity.

This is  $\sim \frac{1}{20}$  flux at  $\Theta$  surface.

$$\text{Now } \sqrt{\omega} K k = \rho g c \sqrt{\frac{\omega}{2K}} \sqrt{\omega} = [\rho c \sqrt{\omega}] \sqrt{\omega}.$$

Thus, eq. (17) shows that, knowing the amplitude of the temperature variation at a given depth, we may derive  $\rho c \sqrt{\omega}$ . This quantity is known as the thermal inertia. Rewriting again, and allowing for the integration const., we have

$$T = T_{oo} + \frac{F_0 e^{-Kz}}{\rho c \sqrt{\omega}} \cos(\omega t - kz - \phi' - \frac{\pi}{4}) \quad (18)$$

if  $\rho c \sqrt{\omega}$  were obtained some other way, eq. (18) could be used to deduce  $F_0$ , the flux at the planetary surface. For  $\Phi$  this quantity would be a blow indeed.

To quote sand, at  $z=5\text{cm}$ ,  $Kz=0.063$ , and  $e^{-Kz} \approx 0.94$ .  
at  $z=10\text{cm}$ ,  $Kz \approx 0.12$ ,  $+e^{-Kz} \approx 0.89$ .

Not quite enough to explain 3 cm amplitude is  $73^\circ$  and 10 cm amplitude is  $31^\circ$ ; but at least in proper direction). Take 3 cm observation.

$$\therefore T_{oo} + \frac{F_0 \cdot 1}{1.4 \times 10^{-2} \sqrt{\omega}} = 73. \quad \text{But } T_{oo} \text{ makes no contribution to the amplitude.}$$

$$\omega = 6.15 \times 10^{-7} \text{ radians/sec. } \sqrt{\omega} = 7.9 \times 10^{-4}.$$

$$\therefore F_0 = 73 \times 1.4 \times 10^{-2} \times 7.9 \times 10^{-4} + T_{oo} = 1.4 \times 10^{-2} + 7.9 \times 10^{-4}$$

$$= 8.0 \times 10^{-4} + 1.1 \times 10^{-5} T_{oo} \text{ erg cm}^{-2} \text{ sec}^{-1}.$$

If this formula is correct, very little light would reach the surface of  $\oplus$ . But it doesn't seem to reduce properly for the  $\oplus$ . Here,  $\omega$  is  $\sim 100$  lyer, & observed flux is 10<sup>-7</sup> lyer.  $\therefore$  expect amplitude for sand to be 10<sup>6</sup> lyer, patently impossible. But for  $\oplus$ , the light capacity of the atmosphere makes a sinusoidal flux, rather than a half sinusoidal  $\cap$  flux more reasonable.

If the surface temperature is any periodic function of the time whatever it can be expressed as a Fourier series. Any periodic function  $T(t)$  can be expressed as

$$T(t) = A_0 + A_1 \cos(\omega t - \phi_1) + A_2 \cos(2\omega t - \phi_2) + \dots$$

where  $2\omega$  is, of course, the first overtone of the fundamental frequency  $\omega$  etc. We want to express in this manner the temperature at depth  $z$ , given by

$$T = T_0 e^{-kz} \cos(\omega t - kz - \phi)$$

where  $k = \sqrt{\frac{\omega}{2\alpha}}$ .  $T(t)$  above is the surface temperature variation.

$$\therefore T = A_0 + \sum_{n=1}^{\infty} A_n e^{-\sqrt{\frac{n\omega}{2\alpha}} z} \cos \left\{ n\omega t - \sqrt{\frac{n\omega}{2\alpha}} z - \phi_n \right\}$$
(19)

Casimir & Drayton remark that (19) is valid only for  $\frac{z}{\sqrt{\frac{\omega}{2\alpha}}} \ll 1$  [so terms are removed].

Now measurements have been performed on the terrestrial subsurface temperature variation. It is found that the diurnal temperature variation is not detected at a depth greater than 3 to 4 feet ( $\approx 1.0$  meters); and the annual temperature variation is not detected below 60-70 feet ( $\approx 20$  meters). Below that depth  $T \neq T(t)$ . I.e., the thermal wave propagates downward in summer in attenuated at the 30 mts level. Similarly in winter the thermal wave propagates upwards.

Now take the surface to be the plane  $z = 0$  with periodic temperature variation given by

$$T = T_0 + \sum_{n=1}^{\infty} T_n \cos(n\omega t - \phi_n) \quad (20)$$

The temperature at depth ( $z$ ) is by (19)

$$T = T_0 + \sum_{n=1}^{\infty} T_n e^{-kz\sqrt{n}} \cos(n\omega t - \sqrt{n} k z - \phi_n) \quad (21)$$

where  $k = \sqrt{\frac{\omega}{2\pi c}}$ . ∵ each partial wave (corresponding to each harmonic frequency) is propagated inwards with individual frequency unchanged. But the amplitudes of the high frequency components decline much more rapidly than the amplitude of the fundamental and first few overtones. I.e., long wavelengths are preferentially propagated. This makes sense in the single picture where at least a wavelength of material is required to stop a given wave train. The depth at which the annual variation is damped by the same factor as the diurnal variation is  $\sqrt{365} \approx 19$  times deeper  $\approx 70\text{ft}/4\text{ft} \approx 17$ ; quite consistent.

On F, there is a diurnal period  $\approx 118^\circ$ , and an annual period  $\approx 250^\circ$ . An evaluation of the annual period requires information presently lacking on the inclination of the F rotation axis to the plane of its orbit.

The classic paper on the reduction of terrestrial temperature measurements is due to Helmholtz (Trans. Roy. Soc. Edin. 22: 405 (1861)). A mean temperature curve for a year was found and harmonically analyzed. From it the temperatures at 2 different depths, were determined, thus:

$$T_1 = T_0' + \sum_{n=1}^{\infty} T_n' \cos(n\omega t - \varphi_n') \text{ at } z_1, \quad (22)$$

$$T_2 = T_0'' + \sum_{n=1}^{\infty} T_n'' \cos(n\omega t - \varphi_n'') \text{ at } z_2 \quad (23)$$

The  $\omega$ 's are, of course, the same in the two cases.  
Now (21) also gives  $T$  at these two depths:

$$T = T_0 + \sum_{n=1}^{\infty} T_n e^{-\sqrt{n'} k z} \cos(n\omega t - \sqrt{n'} k z - \varphi_n). \quad (21)$$

Comparing coefficients between (21), (22), & (23) we have

$$T_0 = T_0' = T_0''.$$

$$T_n' = T_n e^{-\sqrt{n'} k z_1},$$

$$T_n'' = T_n e^{-\sqrt{n'} k z_2}$$

$$\varphi_n' = \sqrt{n'} k z_1 + \varphi_n$$

$$\varphi_n'' = \sqrt{n'} k z_2 + \varphi_n$$

$$\therefore \ln T_n' = \ln T_n - \sqrt{n'} k z_1; \ln T_n'' = \ln T_n - \sqrt{n'} k z_2$$

$$\ln T_n' - \ln T_n'' = -\sqrt{n'} k (z_1 - z_2)$$

$$\therefore \boxed{\frac{\ln T_n' - \ln T_n''}{z_2 - z_1} = \sqrt{n'} k = \sqrt{\frac{n \omega}{2 \kappa}}} \quad (24)$$

Also,

$$\varphi_n' - \varphi_n'' = \sqrt{n'} k (z_1 - z_2)$$

$$\therefore \boxed{\frac{\varphi_n'' - \varphi_n'}{z_2 - z_1} = \sqrt{n'} k = \sqrt{\frac{n \omega}{2 \kappa}}} \quad (25)$$

Thus, for  $\varphi$ , we must Fourier analyse the  $T$  data at the two depths (Mayer's has only the fundamental; Drude's has overtones). Then from either the phase or the amplitude we can, for the same harmonic, determine  $k$ . For the  $\Phi$  Helmholtz

$$\text{A typical value of } K_f = \sqrt{\frac{\omega}{2\kappa}} \text{ is } \frac{7.9 \times 10^{-4}}{\sqrt{2 \times 2 \times 10^{-3}}} = \frac{7.9 \times 10^{-4}}{6.32 \times 10^{-2}} = 1.25 \times 10^{-2} \text{ cm}^{-1}.$$

Redo: Essentially already redone on p. 20 for  $n=1$ , where we found  $\Delta\theta = 50^\circ$  was constant w. drug and  $\Delta\phi$ .

To plasma orientation, take  $\Delta\phi = 17^\circ - 11.7^\circ = 5.3^\circ$ .  
 $= \frac{5.3}{57.3} \times 0.0926 \text{ radians.}$

$$\therefore \frac{9.3 \times 10^{-2}}{5 \times 10} = 1.85 \times 10^{-3}$$

$$\therefore (1.85 \times 10^{-3})^2 = 3.4 \times 10^{-6} = \frac{n \omega}{2\kappa}.$$

$$\therefore \kappa = \frac{6.15 \times 10^{-7}}{6.8 \times 10^{-6}} = 9.05 \times 10^{-2} \text{ cm}^2/\text{sec.}$$

This value is  $\approx$  that for still air and much larger than that for water ( $2 \times 10^{-3} \text{ cm}^2/\text{sec.}$ ). But  $\kappa \propto (\Delta\phi)^2$ .  $\therefore$  a factor 40 in  $\kappa$  requires a factor 6 in  $\Delta\phi$ . Is  $\Delta\phi \approx 45^\circ$  possible?

almost complete agreement between values of  $\kappa$  deduced from amplitude data and from phase data for the first harmonic. Somewhat poorer agreement was found for higher harmonics.

In terrestrial soils the determination of  $\kappa$  from  $T$  measurements is greatly complicated by the presence and non-uniform distribution of water etc., e.g., Keen, The Physical Properties of the Soil - Rothamsted Monographs on Agricultural Science, 1931, Chap. IX) The addition of water to dry soil generally increases the thermal conductivity,  $\kappa$ ; and somewhat increases (factors of 2 and 3) the thermal diffusivity  $\chi$ .

For  $\varphi$ , from (24), and roughly,  $T(3\text{cm}) \approx 80^\circ$ ,  $T(10\text{cm}) \approx 43^\circ$ , and

$$\frac{\ln 80 - \ln 43}{7} = \sqrt{\frac{1}{2} \times 6.15 \times 10^{-7}} + \sqrt{\chi \kappa}$$

$$\begin{aligned}\ln 80 &= 2.303 \log_{10} 80 = 2.303 (1 + \log_{10} 8) = 2.303 + (1.903) = 4.40 \\ \ln 43 &= 2.303 \log_{10} 43 = 2.303 (1 + \log_{10} 4.3) = 2.303 + 1.634 = \frac{3.48}{0.92}\end{aligned}$$

$$\therefore \sqrt{\chi \kappa} = \frac{7}{0.92} \times \sqrt{3.08 \times 10^{-7}} \quad \chi \kappa = 58 \times 3.08 \times 10^{-7} = 1.8 \times 10^{-5} \text{ cm}^2 \text{ sec}^{-1}.$$

This is, of course, 2 orders of magnitude less than granite sand or rocks.

Now phase methods: From P. 16,  $\frac{\Delta \Phi}{\Delta z} = \text{either } \frac{29}{7} = 4.14$  or  $\frac{5}{7} = 0.71$ . This is much less than the  $0.92/7 = 0.13$  given by the amplitude method. For  $\Delta \Phi = 5^\circ$ ,  $\chi \kappa$  is  $5.45 \times 10^{-7} \text{ cm}^2 \text{ sec}^{-1}$ , or  $\chi \kappa = 5.4 \times 10^{-7} \text{ cm}^2 \text{ sec}^{-1}$ . For  $\Delta \Phi = 29^\circ$ ,  $\chi \kappa$  is  $(\frac{4.14}{0.13})^2 = (31.8)^2 = 10^3 \text{ dyn}$ , or  $2 \times 10^{-2} \text{ cm}^2 \text{ sec}^{-1}$ .  $\therefore$  phase method seems to give  $\oplus$  results; amplitude method does not. Why?

first degrees;  
radians!

### B. Thermal Properties of Matter

From "Handbook of Physical Constants," Geol. Soc. Amer. Spec. Paper No. 36, F. Birch, J. F. Schairer, & H. C. Spicer, eds., 1942, section 17.

Thermal conductivities of various substances are given in watts  $\text{cm}^{-1} \text{deg}^{-1}$ . These  $K$ 's can be converted to  $\lambda$ 's by

$$\lambda = K / \rho c$$

if we know  $\rho$  and  $c$ .  $K \approx \sqrt{K_1^2 + K_2^2}$

at  $30^\circ\text{C}$  for garnetite  $K = \sqrt{(887^2 + 3550^2) \times 10^{-6}}$  watts  $\text{cm}^{-1} (\text{K}^\circ)^{-1}$ .

For calcite,  $\text{CaCO}_3$ ,

<u>T</u>	<u>K</u>
$0^\circ\text{C}$	$\sqrt{(40^2 + 35^2)} \times 10^{-6}$
100	$28 \times 10^{-3}$
200	$25 \times 10^{-3}$
300	$22 \times 10^{-3}$
400	$21 \times 10^{-3}$ watts $\text{cm}^{-1} (\text{K}^\circ)^{-1}$ .

For hematite,  $\text{Fe}_2\text{O}_3$  at  $30^\circ\text{C}$ ,  $K_1 = 121 \times 10^{-3}$ ,  $K_2 = 147 \times 10^{-3}$ .

For  $\alpha$ -quartz,  $\text{SiO}_2$ ,

<u>T</u>	<u><math>K_1</math></u>	<u><math>K_2</math></u>
$0^\circ\text{C}$	$114.3 \times 10^{-3}$	$68.2 \times 10^{-3}$ watts $\text{cm}^{-1} (\text{K}^\circ)^{-1}$
100	79.5	49.4
200	63.2	40.6
300	51.5	35.2
400	43.1	31.0

For granite,

$$\begin{array}{c} \underline{T} \\ 100^\circ\text{C} \\ 500^\circ\text{C} \end{array}$$

$$\begin{array}{c} \underline{K} \\ 23.8 \times 10^{-3} \text{ watts cm}^{-2} (\text{K}^\circ)^{-1} \\ 15.9 \times 10^{-3} \leftarrow \end{array}$$

For basalt  $60^\circ\text{C}$

$$\sim 20 \times 10^{-3}$$

For limestone

$$\begin{array}{c} 20^\circ\text{C} \\ 350^\circ\text{C} \end{array}$$

$$\begin{array}{c} 24 \times 10^{-3} \\ 13 \times 10^{-3} \leftarrow \end{array}$$

For dolomite.

$$\begin{array}{c} \sim 20^\circ\text{C} \\ 100^\circ\text{C} \\ 200^\circ\text{C} \end{array}$$

$$\begin{array}{c} 49.8 \times 10^{-3} \\ 38.9 \times 10^{-3} \\ 33.3 \times 10^{-3} \end{array}$$

For serpentines,

$$\sim 20^\circ\text{C}$$

$$27 \times 10^{-3}$$

Increase of  $K$  after soaking in water is characteristically from 11 to 30% for marble, limestone + sandstone.

Increase of  $K$  after compressing dry rock to  $10,000 \text{ lbs/in}^2$  is characteristically from 1 to 30%. If the rock is wet, this same compression increases  $K$  by only 1 to 6%.

Now we construct a table for soil, snow, ice, etc. ignoring  $\rho^{-1}$  in  $\text{cm}^3 \text{ gm}^{-1}$ ; F, moisture % of wet weight, K, and  $\lambda$ : C is the heat capacity per unit mass.

<u>Substance</u>	<u>S</u>	<u>F</u>	<u>K</u> watts/cm <sup>2</sup> (K°) <sup>-1</sup>	<u>H</u> cm <sup>2</sup> sec <sup>-1</sup>	<u>C</u> cm <sup>2</sup> gm <sup>-1</sup>
Fine quartz flour	1.130	—	$1.67 \times 10^{-3}$		
	0.549	2170	22.2		
Coarse quartz powder	0.611	—	3.8		
	0.535	24	11.3		
Pumice	"dry" "wet"		2.5 5		
Quartz sand	0.606	—	2.6	$2.0 \times 10^{-3}$	2.6
	0.572	8.3	5.9	$3.4 \times 10^{-3}$	5.9
Sandy clay	0.561	15	9.2	$3.8 \times 10^{-3}$	1.4
Saturated cal- careous earth	0.60	43	7.1	$1.9 \times 10^{-3}$	2.2
Densely packed snow	1.85		4.6	$4.1 \times 10^{-3}$	2.1
Ice 0°C -130°C			22.3 40.2		
Sandy soil				$9 \times 10^{-3}$	

For graphite,

$$\frac{T}{0^\circ\text{C}}$$

500

1000

$$\frac{K}{1.68 \text{ watts/cm}^2 (\text{K}^\circ)^{-1}}$$

[no  $\times 10^{-3}$  here!]

0.92

0.55

For S,  $K \approx 2 \times 10^{-3}$  watts/cm<sup>2</sup>(K°)<sup>-1</sup> over a wide range of T.

For H<sub>2</sub>O, I K at 1 atm.

$$0^\circ\text{C}$$

$$5.52 \times 10^{-3}$$

$$80^\circ\text{C}$$

$$6.86 \times 10^{-3}$$

$K = K_0(1+\alpha P)$  for  $P$  in kg/cm<sup>2</sup> + 1  $\leq P \leq 12,000$  ~~kg/cm<sup>2</sup>~~  
 where  $K_0$  is conductivity at  $P=1$  atm.  $\alpha$  ranges from  $1 \times 10^{-6}$  for limonite at  $30^\circ\text{C}$  to  $3.6 \times 10^{-6}$  for rock salt at  $30^\circ\text{C}$ . Since  $\alpha$  increases w. T [at  $75^\circ\text{C}$ ,  $\alpha = 6.7 \times 10^{-6}$ ].

### C. Plane Electromagnetic Waves

Refs.: J. D. Jackson "Classical Electrodynamics," Vol. 2, 1962; chap. 7; and W. K. H. Panofsky and M. Phillips "Classical Electricity & Magnetism" Addison-Wesley, 1955, §§ 11-5 + 11-6.

#### 1. Plane waves in a nonconducting medium

We now derive the fact that transverse plane electromagnetic waves are propagated in nonconducting media in which the dielectric const. + magnetic permeability are const. in the medium. Maxwell's eqs. give

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{B} - \frac{\mu \epsilon}{c} \frac{\partial \vec{E}}{\partial t} &= 0. \end{aligned} \quad \left. \right\} (26)$$

The zeros in the first two Maxwell eqs. exhibit no charges + no currents, i.e., a nonconducting medium.  
Now

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \\ &= -\frac{\mu \epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = -\frac{\mu \epsilon}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) \\ &= +\frac{\mu \epsilon}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}. \end{aligned}$$

Thus both eqs. have the wave eq. form

$$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0$$

I.e., the absolute values of the  $\vec{E}$  and  $\vec{B}$  vectors should be propagated with a phase velocity

$$v = \frac{c}{\sqrt{\mu \epsilon}}$$

(27)

Thus, when  $\mu = 0$ , the index of refraction in a dielectric is given by  $n = \sqrt{\epsilon}$ .

Now in the  $x$ -direction, the wave equation has the solution

$$u = e^{ik \cdot \vec{x} - i\omega t} \quad (28)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2} = -k^2 u$$

$$\frac{\partial^2 u}{\partial x^2} = -\omega^2 u.$$

$$\therefore -k^2 u + \frac{\omega^2}{v^2} u = 0. \quad \therefore k^2 = \omega^2/v^2, \text{ and}$$

$$k = \frac{\omega}{v} = \pm \sqrt{\mu \epsilon} \frac{\omega}{c} \quad (29)$$

Here  $\omega$  is the frequency of the wave and  $k = |\vec{k}|$  is the magnitude of the wave vector. For plane waves propagating in the  $x$ -direction, the fundamental solution is the sum of the solns implied in (29).

$$\begin{aligned} u(x, t) &= Ae^{ikx - i\omega t} + Be^{-ikx - i\omega t} \\ &= Ae^{i(kx - \omega t)} + Be^{-i(kx - \omega t)} \\ &= Ae^{i\kappa(x - vt)} + Be^{-i\kappa(x + vt)} \end{aligned} \quad (30)$$

Now the Fourier integral theorem states that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k) e^{ikx} dk$$

where

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx.$$

It follows, according to Jackson, that by linear superposition, a gen. sol. of the wave eq. is

$$u(x, t) = f(x - vt) + g(x + vt) \quad (31)$$

where  $f$  +  $g$  are arbitrary functions. Is this a sol. of the wave eq.?

$$\begin{aligned} \nabla^2 u &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} \\ \frac{\partial^2 u}{\partial t^2} &= \left[ \frac{\partial^2 f}{\partial x^2} v^2 + \frac{\partial^2 g}{\partial x^2} v^2 \right] \\ \therefore \nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} &\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} - \left[ \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 g}{\partial x^2} \right] v^2 = 0. \end{aligned}$$

Let  $x \pm vt = \xi$

$$\begin{aligned} \nabla^2 u &= \frac{\partial}{\partial \xi} \left[ \frac{\partial f}{\partial \xi} \frac{\partial g}{\partial x} \right] + \frac{\partial}{\partial \xi} \left[ \frac{\partial g}{\partial \xi} \frac{\partial f}{\partial x} \right] \\ &= \frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 g}{\partial \xi^2} \\ \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial \xi} \left[ \frac{\partial f}{\partial \xi} \frac{\partial g}{\partial x} \right] + \frac{\partial}{\partial \xi} \left[ \frac{\partial g}{\partial \xi} \frac{\partial f}{\partial x} \right] \\ &= \frac{\partial}{\partial x} \left[ -v \frac{\partial f}{\partial \xi} \right] + \frac{\partial}{\partial x} \left[ +v \frac{\partial g}{\partial \xi} \right] \\ &= -v \frac{\partial}{\partial \xi} \left[ \frac{\partial f}{\partial \xi} \right] \frac{\partial \xi}{\partial x} + v \frac{\partial}{\partial \xi} \left[ \frac{\partial g}{\partial \xi} \right] \frac{\partial \xi}{\partial x} \\ &= +v^2 \frac{\partial^2 f}{\partial \xi^2} + v^2 \frac{\partial^2 g}{\partial \xi^2} \\ \therefore \nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} &\Rightarrow \frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 g}{\partial \xi^2} - \frac{\partial^2 f}{\partial \xi^2} - \frac{\partial^2 g}{\partial \xi^2} = 0 \end{aligned}$$

as it should. Eq. (31) represent waves travelling along the  $x$ -axis, to the right and to the left.

Having specified the wave nature of the eqs., we must still worry about the vector nature of the fields. We adopt the convention  $\vec{E}$  +  $\vec{B}$  are obtained by taking the real part of the appropriate complex quantities:

$$\begin{aligned} \vec{E}(x, t) &= \hat{e}_1 E_0 e^{i \vec{k} \cdot \vec{x} - i \omega t} \\ \vec{B}(x, t) &= \hat{e}_2 B_0 e^{i \vec{k} \cdot \vec{x} - i \omega t} \end{aligned} \quad \} \quad (32)$$

where  $\hat{e}_1$  and  $\hat{e}_2$  are const. unit vectors, and  $E_0$  +  $B_0$  are complex amplitudes, also ~~const~~ constant in space + time.

$\vec{\nabla} \cdot \vec{E} = 0$  and  $\vec{\nabla} \cdot \vec{B} = 0$  demand, since  $\vec{\nabla}$  is max in the  $\vec{k}$ -direction,

$$\hat{i}_1 \cdot \vec{k} = 0, \text{ and } \hat{i}_2 \cdot \vec{k} = 0. \quad (33)$$

Therefore, both  $\vec{E}$  and  $\vec{B}$  are perpendicular to the direction of propagation  $\vec{k}$ . Such a wave is called a transverse wave.

Substituting (32) into

$$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$$

gives

$$E_0 \vec{\nabla} \times \hat{i}_1 e^{i\vec{k} \cdot \vec{x} - i\omega t} + \frac{1}{c} \hat{i}_2 B_0 \frac{\partial}{\partial t} e^{i\vec{k} \cdot \vec{x} - i\omega t} = 0$$

$$\therefore E_0 i \vec{k} \times \hat{i}_1 e^{i\vec{k} \cdot \vec{x} - i\omega t} - \frac{i\omega}{c} \hat{i}_2 B_0 e^{i\vec{k} \cdot \vec{x} - i\omega t} = 0$$

$$\therefore i \left[ (\vec{k} \times \hat{i}_1) E_0 - \frac{\omega}{c} \hat{i}_2 B_0 \right] e^{i\vec{k} \cdot \vec{x} - i\omega t} = 0.$$

$$\therefore (\vec{k} \times \hat{i}_1) E_0 = \frac{\omega}{c} \hat{i}_2 B_0.$$

$\therefore$  for direction,

$$\hat{i}_2 = \vec{k} \times \hat{i}_1. \quad (34)$$

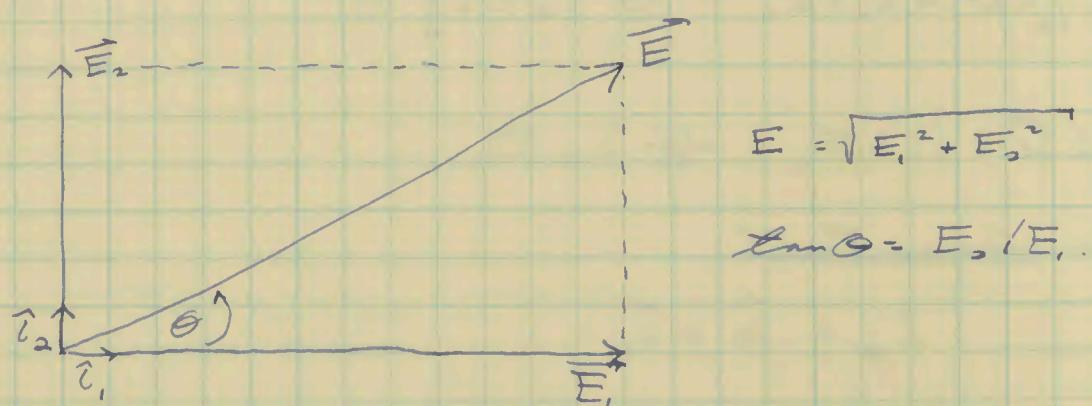
and for magnitude,

$$k E_0 = \frac{\omega}{c} B_0.$$

$$\text{Since from (29) } k = \sqrt{\mu\epsilon} \frac{\omega}{c},$$

$$B_0 = \sqrt{\mu\epsilon} E_0 \quad (35)$$

$(\hat{i}_1, \hat{i}_2, \vec{k})$  is a set of mutually orthogonal vectors,  $\vec{E}$  and  $\vec{B}$  have a constant ratio, and,  $\vec{E}$  and  $\vec{B}$  are in phase.



$$E = \sqrt{E_1^2 + E_2^2}$$

$$\tan \theta = E_2 / E_1$$

The plane waves of eq. (32) have  $\vec{E} \propto \hat{\ell}_1$ . Thus it is linearly polarized with polarization vector  $\hat{\ell}_1$ . To describe a more general state of polarization, we need another linearly polarized wave, independent of the first.  $\therefore$  take

$$\begin{aligned}\vec{E}_1 &= \hat{\ell}_1 E_1 e^{i\vec{k} \cdot \vec{x} - i\omega t} \\ \vec{E}_2 &= \hat{\ell}_2 E_2 e^{i\vec{k} \cdot \vec{x} - i\omega t}\end{aligned}\quad \left. \right\} \quad (36)$$

Each is a linearly independent solution of the wave equation. From (34) and (35), the corresponding  $\vec{B}$ 's are

$$\begin{aligned}\vec{B}_1 &= \sqrt{\mu\epsilon} \hat{k} \times \vec{E}_1 \\ \vec{B}_2 &= \sqrt{\mu\epsilon} \hat{k} \times \vec{E}_2\end{aligned}\quad \left. \right\} \quad (37)$$

Thus the  $\vec{B}$ 's are  $\perp$  to the propagation vector and to their respective  $\vec{E}$ -vectors.

The amplitude  $E_1$  and  $E_2$  of (36) are complex numbers, to allow for the possibility of a phase difference between the waves (leading to elliptical polarization).

A general solution for a plane wave propagating in the direction  $\hat{k}$  is given by a linear combination of  $\vec{E}_1$  and  $\vec{E}_2$ :

$$\vec{E}(\vec{x}, t) = (\hat{\ell}_1 E_1 + \hat{\ell}_2 E_2) e^{i\vec{k} \cdot \vec{x} - i\omega t} \quad (38)$$

If  $E_1$  and  $E_2$  have the same phase, then (38) represents a linearly polarized wave. Its polarization vector makes an angle

$$\theta = \arctan(E_2/E_1)$$

with  $\hat{\ell}_1$ . Its magnitude is  $\sqrt{E_1^2 + E_2^2}$

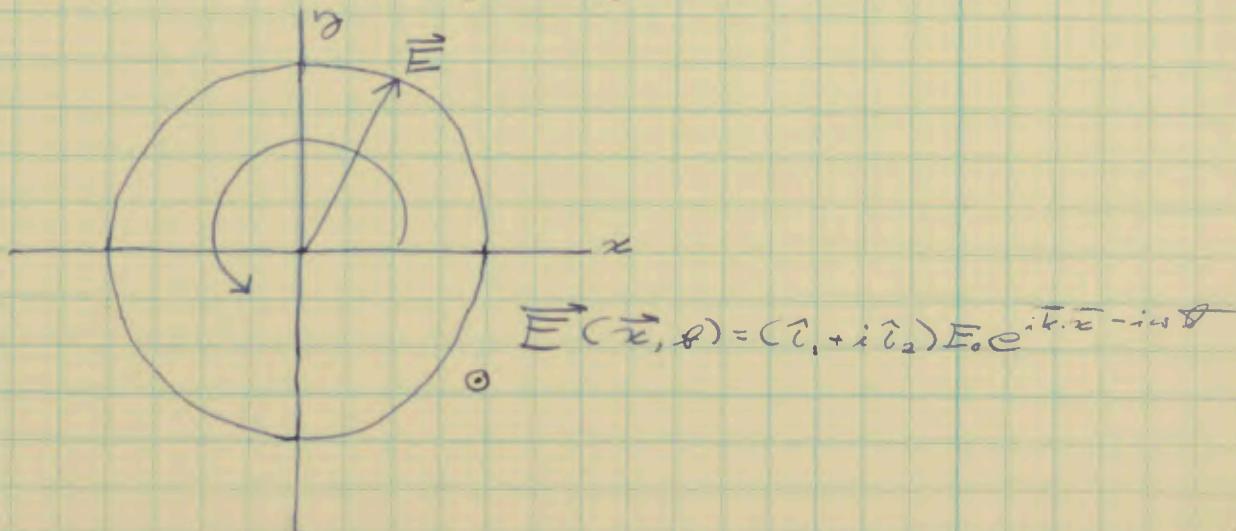
If  $E_1$  and  $E_2$  have different phases, then the wave represented by eq. (38) is elliptically polarized. A special case of elliptical polarization is circular polarization. Then  $E_1$  and  $E_2$  have the same magnitude, but differ in phase by  $90^\circ$ . We then have

$$\vec{E}(\vec{x}, t) = E_0 (\hat{i}_1 + i \hat{i}_2) e^{i \vec{k} \cdot \vec{x} - i \omega t} \quad (39)$$

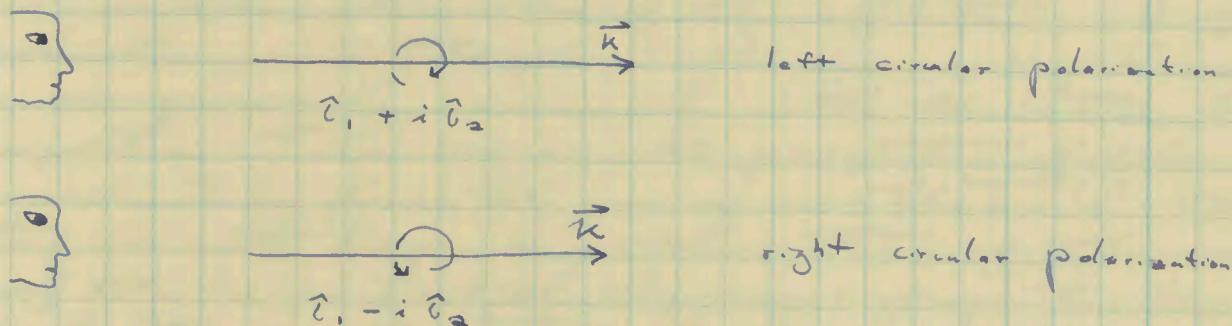
with  $E_0$  as the common real amplitude. Diagrams were shown so that the wave is propagating in the positive  $z$ -direction,  $\vec{k} = \hat{k}_3 = \hat{z}$ , while  $\hat{i}_1$  and  $\hat{i}_2$  are in the  $x$ - and  $y$ -directions respectively. Then the components of the actual electric field, obtained by taking the real part of eq. (37), are:

$$\begin{aligned} \vec{E}_x(\vec{x}, t) &= E_0 \operatorname{Re} (\hat{i}_1 + i \hat{i}_2) e^{i \vec{k} \cdot \vec{x} - i \omega t} \\ &= E_0 (\hat{i}_1 \cos(kz - \omega t) + i \sin(kz - \omega t)) \\ &= E_0 \hat{i}_1 \cos(kz - \omega t) + E_0 \hat{i}_2 \sin(kz - \omega t) \\ \therefore \vec{E}_x(\vec{x}, t) &= E_0 \cos(kz - \omega t) \quad \} \\ \cdot \vec{E}_y(\vec{x}, t) &= \mp E_0 \sin(kz - \omega t), \quad \} \end{aligned} \quad (40)$$

thus demonstrating the desired  $90^\circ$  phase difference explicitly. At a fixed point in space, these fields are const. in magnitude, but sweep around in a circle at frequency  $\omega$ :



In  $(\hat{e}_1 \pm i \hat{e}_2)$ , the + sign indicates clockwise rotation when the observer is looking in the direction the wave is going, and is called left circularly polarized in optics. A right circularly polarized wave has a - sign, and shows clockwise rotation when the observer is facing the oncoming wave.

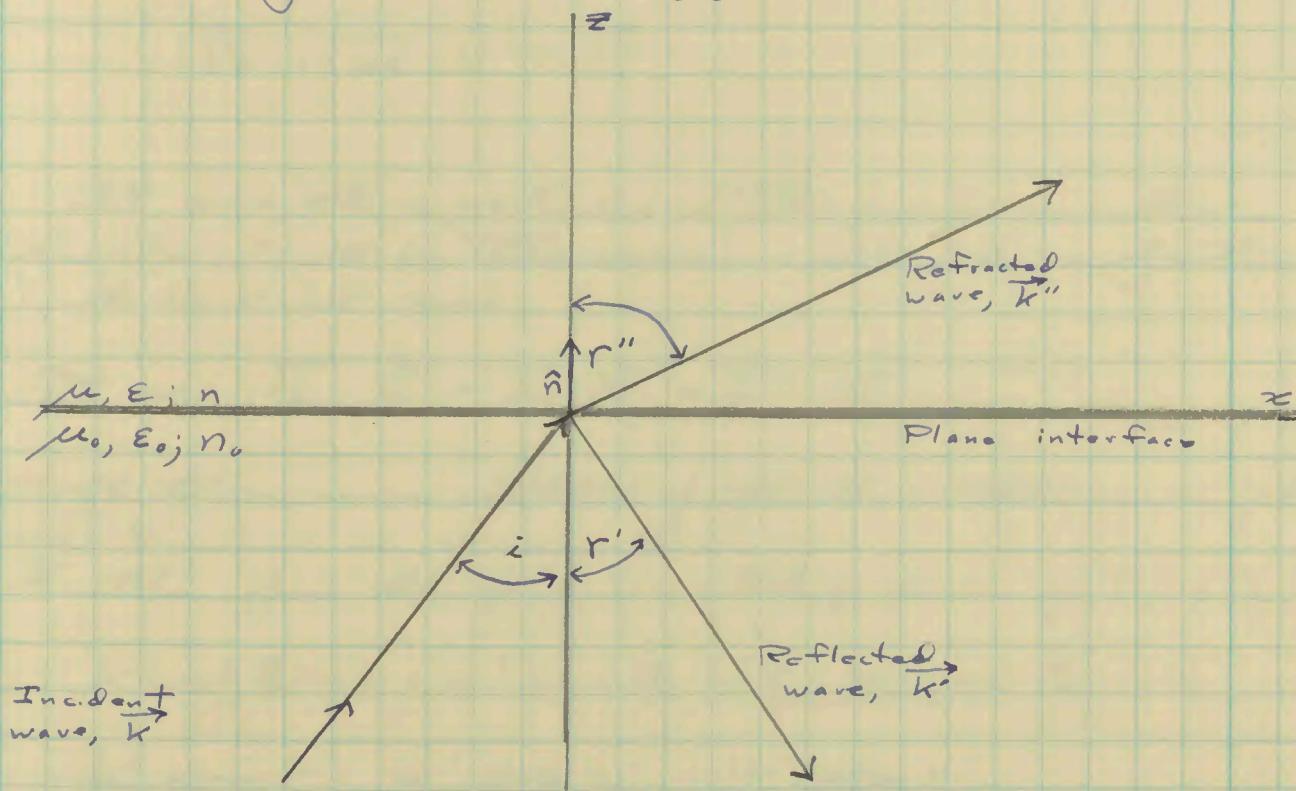


The radio "handedness" convention is opposite from the optical convention.

## D. Reflection and Refraction of Electromagnetic Waves at a Plane Interface between Dielectrics

This phenomenon involves kinematic properties such as Snell's law and the equality between the angles of incidence & reflection, and dynamic properties such as the intensities of the reflected and refracted radiation, and the phase changes and polarization. We are primarily concerned with the latter dynamic properties. The kinematic properties follow immediately from the wave eq. and the b.c.'s. But they do not depend on the nature of the waves or of the b.c.'s. But the dynamic properties depend explicitly on the nature of the e.m. fields and the b.c.'s.

The geometry is as ff.:



$\hat{n}$  is the unit vector normal to the  $x-y$  plane.

From (36) and (37), the three waves can be written as follows:

Incident:

$$\vec{E} = \vec{E}_0 e^{i\vec{k} \cdot \vec{x} - i\omega t}$$

$$\vec{B} = \sqrt{\mu_0 \epsilon_0} \hat{k} \times \vec{E}$$

Reflected:

$$\vec{E}' = \vec{E}'_0 e^{i\vec{k}' \cdot \vec{x} - i\omega t}$$

$$\vec{B}' = \sqrt{\mu_0 \epsilon_0} \hat{k}' \times \vec{E}'$$

Refracted:

$$\vec{E}'' = \vec{E}''_0 e^{i\vec{k}'' \cdot \vec{x} - i\omega t}$$

$$\vec{B}'' = \sqrt{\mu_0 \epsilon_0} \hat{k}'' \times \vec{E}''$$

At  $x=0$ , the spatial and temporal variations of all fields must be the same.  $\therefore$  all phase factors must be equal at  $x=0$ .

$$\text{At } x=0, \vec{k} \cdot \vec{x} = \vec{k}' \cdot \vec{x} = \vec{k}'' \cdot \vec{x}.$$

$\therefore$  all three wave vectors must lie in the same plane, (why? Projections on  $x$ -axis all =?)

Further, from the figure on the previous p. (p. 51),

$$k \cos(\frac{\pi}{2} - i) = k' \cos(\frac{\pi}{2} - r') = k'' \cos(\frac{\pi}{2} - r'')$$

$$\therefore k \sin i = k' \sin r' = k'' \sin r''.$$

But  $k = k'$ .

$$\therefore i = r',$$

(41)

and the reflector is specular

Similarly, we find

$$k \sin i = k'' \sin r'' \quad (42)$$

But the wave vectors have magnitudes

$$|\vec{k}| = |\vec{k}'| = k = \sqrt{\mu_0 \epsilon_0} \frac{\omega}{c}$$

from eq. (29), p. 39, derived directly from the sol. to the wave eq. Also,

$$|\vec{k}''| = k'' = \sqrt{\mu_0 \epsilon_0} \frac{\omega}{c}.$$

$\therefore$  (42) gives

$$\frac{\sin i}{\sin r''} = \frac{k''}{k} = \frac{\sqrt{\mu_0 \epsilon_0} \frac{\omega}{c}}{\sqrt{\mu_0 \epsilon_0} \frac{\omega}{c}} = \frac{n}{n_0}$$

$$\therefore n_0 \sin i = n \sin r'' \quad (43)$$

which is Snell's law.

Now the dynamic properties are contained in the b.c.'s expressing the continuity of

$$\vec{D}_\perp \text{ and } \vec{B}_\perp$$

(normal components)

} (44)

and of

$$\vec{E}_\parallel \text{ and } \vec{H}_\parallel$$

(tangential components)

} (45)

at the boundary. These continuity relations can be obtained for  $\vec{D}_\perp$  and  $\vec{B}_\perp$  by cutting a Gaussian pillbox at the boundary, applying integrating, e.g.,  $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$  over the volume, applying Gauss' theorem, and contracting the volume to zero. For  $\vec{E}_\parallel + \vec{H}_\parallel$ , we cut a rectangle at the boundary, integrate, e.g.,  $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$  over the area, applying Stoke's theorem, and contracting the area to zero.

At  $x=0$ , these b.c.'s give, for the fields on p. 53,

$$D_{\perp} + D'_{\perp} = D''_{\perp}, \text{ or}$$

$$[\epsilon_0(\vec{E}_0 + \vec{E}') - \epsilon \vec{E}''] \cdot \hat{n} = 0. \quad (45)$$

$$B_{\perp} + B'_{\perp} - B''_{\perp} = 0, \text{ or}$$

$$[\sqrt{\mu_0 \epsilon_0} (\hat{k} \times \vec{E}_0 + \hat{k}' \times \vec{E}') - \sqrt{\mu \epsilon} \hat{k}'' \times \vec{E}''] \cdot \hat{n} = 0. \quad (46)$$

$$E_{\parallel} + E'_{\parallel} - E''_{\parallel} = 0, \text{ or}$$

$$[\vec{E}_0 + \vec{E}' - \vec{E}''] \times \hat{n} = 0. \quad (47)$$

$$H_{\parallel} + H'_{\parallel} - H''_{\parallel} = 0, \text{ or}$$

$$[\sqrt{\frac{\epsilon_0}{\mu_0}} (\hat{k} \times \vec{E}_0 + \hat{k}' \times \vec{E}') - \sqrt{\frac{\epsilon}{\mu}} \hat{k}'' \times \vec{E}''] \times \hat{n} = 0. \quad (48)$$

Since  $|\vec{k}| = \sqrt{\mu_0 \epsilon_0} \frac{\omega}{c}$ , substitution of  $\vec{k}$ 's for unit vector  $\hat{k}$ 's removes the  $\sqrt{\mu_0 \epsilon_0}$  factor from eqs. (46) + (48), since the  $\omega/c$  factor is common to all terms. Renaming eqs. (45) - (48), then, we have

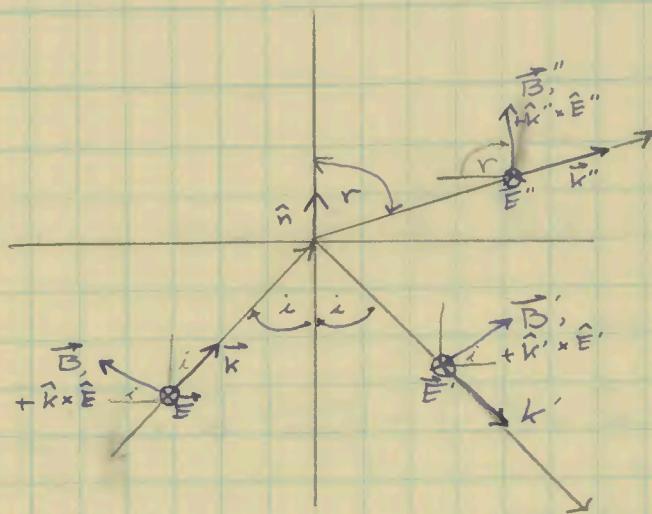
$$[\epsilon_0(\vec{E}_0 + \vec{E}') - \epsilon \vec{E}''] \cdot \hat{n} = 0 \quad (49)$$

$$[\vec{k} \times \vec{E}_0 + \vec{k}' \times \vec{E}' - \vec{k}'' \times \vec{E}''] \cdot \hat{n} = 0 \quad (50)$$

$$[\vec{E}_0 + \vec{E}' - \vec{E}''] \times \hat{n} = 0. \quad (51)$$

$$[\frac{1}{\mu_0}(\vec{k} \times \vec{E}_0 + \vec{k}' \times \vec{E}') - \frac{1}{\mu} \vec{k}'' \times \vec{E}''] \times \hat{n} = 0. \quad (52)$$

Now we consider two separate situations, one in which the incident plane wave is linearly polarized with its polarization vector  $\hat{i}$  (or its electric vector  $\vec{E}$ ) is in the plane of incidence ( $\hat{k} \times \hat{n}$  plane), and the other when  $\vec{E}$  is  $\perp$  to the  $\hat{k} \times \hat{n}$  plane. The general case of arbitrary elliptic polarization can be obtained, as before, by an appropriate linear combination of the two results, using complex amplitudes.



$\vec{E}$ -vectors into the page.  
 $\vec{B}$  directions chosen so  
that  $\hat{S} = \hat{k}$ , where  
 $\hat{S} = \vec{B}/|\vec{B}|$ , and  $\hat{S}$  is  
the Poynting flux.

First consider the case of  $\vec{E} \perp$  to the  $\vec{k} \times \hat{n}$  plane. Then, the  $\vec{B}$ 's are in the plane of incidence.  $\therefore$  all the  $\vec{E}$ 's are  $\perp$  to  $\hat{n}$  and eq. (49) is identically zero. Eq. (51) gives

$$E_0 + E_0' - E_0'' = 0. \quad (53)$$

Eq. (50), or, better, (48) gives, e.g., for the first term,

$\vec{k} \times \vec{E}_0$  is, by the r.h. rule,  $\parallel \vec{B}$  as shown on the adjoining page. The component remaining after  $(\vec{k} \times \vec{E}_0) \times \hat{n}$  is then  $\perp \hat{n}$ , i.e., the projection  $E_0 \cos i$ .

$$\therefore \sqrt{\frac{\epsilon_0}{\mu_0}} (E_0 - E_0') \cos i - \sqrt{\frac{\epsilon}{\mu}} E_0'' \cos r = 0. \quad (54)$$

Eq. (50), + Snell law, only duplicates this result.

$$\therefore \sqrt{\frac{\epsilon_0}{\mu_0}} (E_0 - E_0') \cos i - \sqrt{\frac{\epsilon}{\mu}} (E_0 + E_0') \cos r = 0.$$

$$E_0 (\sqrt{\frac{\epsilon_0}{\mu_0}} \cos i - \sqrt{\frac{\epsilon}{\mu}} \cos r) - E_0' (\sqrt{\frac{\epsilon_0}{\mu_0}} \cos i + \sqrt{\frac{\epsilon}{\mu}} \cos r) = 0$$

$$\therefore \frac{E_0'}{E_0} = \frac{\sqrt{\frac{\epsilon_0}{\mu_0}} \cos i - \sqrt{\frac{\epsilon}{\mu}} \cos r}{\sqrt{\frac{\epsilon_0}{\mu_0}} \cos i + \sqrt{\frac{\epsilon}{\mu}} \cos r}$$

Setting  $\mu = \mu_0$ , as is generally true for optical frequencies we obtain

$$\frac{E_0'}{E_0} = \frac{\sqrt{\epsilon_0} \cos i - \sqrt{\epsilon} \sqrt{1 - \sin^2 r}}{\sqrt{\epsilon_0} \cos i + \sqrt{\epsilon} \sqrt{1 - \sin^2 r}}$$

↑  
but not necessarily  
for radio frequencies

$$\sqrt{\epsilon_0} \sin i = \sqrt{\epsilon} \sin r.$$

$$\therefore \frac{E_0'}{E_0} = \frac{\sqrt{\epsilon_0} \cos i - \sqrt{\epsilon - \epsilon_0 \sin^2 i}}{\sqrt{\epsilon_0} \cos i + \sqrt{\epsilon - \epsilon_0 \sin^2 i}}, \text{ or}$$

$$\frac{E_0'}{E_0} = \frac{\cos i - \sqrt{\frac{\epsilon}{\epsilon_0} - \sin^2 i}}{\cos i + \sqrt{\frac{\epsilon}{\epsilon_0} - \sin^2 i}}$$

(55)

for reflection.

For refraction, we eliminate  $E'$  from eqs. (53) + (54):

$$E'_r = E''_r - E_r.$$

$$\therefore \sqrt{\frac{\epsilon_0}{\mu_0}} (E_r - E''_r + E_r) \cos i - \sqrt{\frac{\epsilon}{\mu}} E''_r \cos r = 0.$$

$$2E_r \sqrt{\frac{\epsilon_0}{\mu_0}} \cos i = E''_r (\sqrt{\frac{\epsilon}{\mu}} \cos r + \sqrt{\frac{\epsilon_0}{\mu_0}} \cos i)$$

$$\therefore \frac{E''_r}{E_r} = \frac{2 \sqrt{\frac{\epsilon_0}{\mu_0}} \cos i}{\sqrt{\frac{\epsilon}{\mu}} \cos r + \sqrt{\frac{\epsilon_0}{\mu_0}} \cos i}$$

Again,  $\mu = \mu_0$ ,

$$\frac{E''_r}{E_r} = \frac{2 \sqrt{\epsilon_0} \cos i}{\sqrt{\epsilon} \sqrt{1 - \sin^2 r} + \sqrt{\epsilon_0} \cos i}$$

$$\sqrt{\epsilon_0} \sin i = \sqrt{\epsilon} \sin r$$

$$\therefore \frac{E''_r}{E_r} = \frac{2 \sqrt{\epsilon_0} \cos i}{\sqrt{\epsilon - \epsilon_0 \sin^2 i} + \sqrt{\epsilon_0} \cos i}$$

$$\therefore \frac{E''_r}{E_r} = \frac{2 \cos i}{\sqrt{\frac{\epsilon_0}{\epsilon} - \sin^2 i} + \cos i} \quad (56)$$

$$= \frac{2}{1 + \sqrt{\frac{\epsilon_0}{\epsilon} \sec^2 i - \tan^2 i}}$$

for refraction.

Now consider the case that  $\vec{E}$  is "to the plane of incidence".

In just the same way that (53) + (54) were derived, using a diagram like that on p. 58, we find

$$\cos i (E_0 - E_0') - \cos r E_0'' = 0 \quad (57)$$

and

$$\sqrt{\frac{\epsilon_0}{\mu_0}} (E_0 + E_0') - \sqrt{\frac{\epsilon}{\mu}} E_0'' = 0. \quad (58)$$

Eq. (50) is identically zero, and (49) + Snell's law only implies eq. (58). Setting  $\mu_0 = \mu$  for the moment, we have

$$E_0'' = \sqrt{\frac{\epsilon_0}{\epsilon}} (E_0 + E_0')$$

$$\therefore \cos i (E_0 - E_0') - \sqrt{\frac{\epsilon_0}{\epsilon}} \cos r (E_0 + E_0') = 0.$$

$$\therefore E_0 (\cos i - \sqrt{\frac{\epsilon_0}{\epsilon}} \cos r) - E_0' (\cos i + \sqrt{\frac{\epsilon_0}{\epsilon}} \cos r) = 0.$$

$$\begin{aligned} \therefore \frac{E_0'}{E_0} &= \frac{\cos i - \sqrt{\frac{\epsilon_0}{\epsilon}} \cos r}{\cos i + \sqrt{\frac{\epsilon_0}{\epsilon}} \cos r} \\ &= \frac{\cos i - \sqrt{\frac{\epsilon_0}{\epsilon}} \sqrt{1 - \sin^2 r}}{\cos i + \sqrt{\frac{\epsilon_0}{\epsilon}} \sqrt{1 - \sin^2 r}} \end{aligned}$$

$$\sqrt{\epsilon_0} \sin i = \sqrt{\epsilon} \sin r$$

$$\therefore \frac{E_0'}{E_0} = \frac{\cos i - \sqrt{\frac{\epsilon_0}{\epsilon}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2 i}}{\cos i + \sqrt{\frac{\epsilon_0}{\epsilon}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2 i}}$$

Divide numerator & denominator by  $\sqrt{\frac{\epsilon_0}{\epsilon}}$ ; i.e., multiply by  $\frac{\sqrt{\epsilon_0}}{\sqrt{\epsilon}}$ .

$$\therefore \frac{E_0'}{E_0} = \frac{\frac{\epsilon_{0n}}{\epsilon_{en}} - \sqrt{\frac{\epsilon_{0n}}{\epsilon_{en}} - \sin^2 i}}{\frac{\epsilon_{0n}}{\epsilon_{en}} + \sqrt{\frac{\epsilon_{0n}}{\epsilon_{en}} - \sin^2 i}} \quad (59)$$

for reflection

For refraction, we eliminate  $E_0'$  from eqs. (57) & (58).

$$E_0 - E_0' = \frac{\cos r}{\cos i} E_0''$$

$$\therefore E_0' = E_0 - \frac{\cos r}{\cos i} E_0''$$

$$\therefore \sqrt{\frac{\epsilon_0}{\mu_0}} (E_0 + E_0 - \frac{\cos r}{\cos i} E_0'') - \sqrt{\frac{\epsilon}{\mu}} E_0'' = 0.$$

$$\therefore 2\sqrt{\frac{\epsilon_0}{\mu_0}} E_0 = \sqrt{\frac{\epsilon}{\mu}} E_0'' + \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\cos r}{\cos i} E_0''$$

$$\therefore \frac{E_0''}{E_0} = \frac{2\sqrt{\frac{\epsilon_0}{\mu_0}}}{\sqrt{\frac{\epsilon}{\mu}} + \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\cos r}{\cos i}}$$

$$= \frac{2\sqrt{\epsilon_0}}{\sqrt{\epsilon} + \sqrt{\epsilon_0} \frac{1 - \sin^2 r}{\cos i}}$$

$$\approx \frac{2}{\sqrt{\frac{\epsilon_0'}{\epsilon_0}} + \frac{1 - \frac{\epsilon_0}{\epsilon} \sin^2 i}{\cos i}}$$

$$= \frac{2\sqrt{\epsilon_0}}{\sqrt{\epsilon} + \sqrt{\epsilon_0} (\sec i - \frac{\epsilon_0}{\epsilon} \sin i \tan i)}$$

$$= \frac{2 \cos i}{\sqrt{\frac{\epsilon}{\epsilon_0}} \cos i + 1 - \frac{\epsilon_0}{\epsilon} \sin^2 i}$$

$$= \frac{2 \cos i}{\sqrt{\frac{\epsilon}{\epsilon_0}} \cos i + 1 - \frac{\epsilon_0}{\epsilon} (1 - \cos^2 i)}$$

$$\frac{E_0''}{E_0} = \frac{2 \cos i}{1 - \frac{\epsilon_0}{\epsilon} + \sqrt{\frac{\epsilon}{\epsilon_0}} \cos i + \frac{\epsilon_0}{\epsilon} \cos^2 i} \quad (60)$$

for refraction

Note for future reference that, since  $\sin(-\varphi) = -\sin\varphi$ ,  
 that replacing  $i$  by  $-i$  + vice versa does not affect  
 $(E'_0/E_0)^2$  for either case. But  $(E''_0/E_0)^2$  is affected.  
 $\therefore R$  is invariant to an  $i$  interchange.  $T$  is not.  
 But how can this be if  $T = 1 - R$ ?

Setting  $\mu = \mu_0 = \epsilon_0 = 1$ , and  $\epsilon = \sqrt{n} > 1$ , we see that both  
 the numerator + the denominators of  $R_{||}$  are greater than  
 the corresponding values for  $R_{\perp}$ . In this fraction numerator  
 $\leq$  denominator. I.e.,  $R_{||} : [p/q]^2$ ,  $R_{\perp} : [p-s/q-s]^2$ .

$$\therefore (R_{||}/R_{\perp})^{1/2} = \frac{p/q}{(p-s/q-s)} = \frac{p(q-s)}{q(p-s)} = \frac{pq - ps}{pq - qs} \quad q \geq p.$$

$\therefore R_{||}/R_{\perp} \geq 1$ . Alternatively, from (61) + (63),  $R_{||} \geq R_{\perp}$ .

$$R_{||}/R_{\perp} = \frac{\cos(i+\varphi)}{\cos(i-\varphi)} \leq 1. \text{ Which is right?}$$

In the case  $\mu = \mu_0$ , and leaving the expression in terms of  $i$  and  $r$ , we find

$$\vec{E} \cdot (\vec{k} \times \hat{n}) = 0 : \quad (\perp \text{ plane})$$

$$\frac{E_0'}{E_0} = - \frac{\sin(i-r)}{\sin(i+r)} \quad (61)$$

$$\frac{E_0''}{E_0} = \frac{2 \cos i \sin r}{\sin(i+r)} \quad (62)$$

$$\vec{E} \times (\vec{k} \times \hat{n}) = 0 \quad (\parallel \text{ plane})$$

$$\frac{E_0'}{E_0} = \frac{2 \cos i \sin r}{\tan(i+r) - \tan(i-r)} \quad (63)$$

$$\frac{E_0''}{E_0} = \frac{2 \cos i \sin r}{\sin(i+r) \cos(i-r)} \quad (64)$$

Eqs. (61) - (64) are known collectively as Fresnel's equations. In planetary problems it is convenient to use only the  $\mathcal{F}$  of incidence,  $i$ , or, as sometime written,  $\theta$ . The Poynting flux  $S = \frac{1}{4\pi} \vec{E} \times \vec{B}$  is proportional to  $E^2$ . The reflected flux is the astronomically determinable quantity. From Eqs. (55) + (59), the energy reflectivity for  $\vec{E} \parallel \perp$  to the plane of incidence is

$$R_{\parallel}(i) = \left[ \frac{\frac{E_{0\parallel}}{E_{0\perp}} - \sqrt{\frac{E_{0\parallel}}{E_{0\perp}} - \sin^2 i}}{\frac{E_{0\parallel}}{E_{0\perp}} + \sqrt{\frac{E_{0\parallel}}{E_{0\perp}} - \sin^2 i}} \right]^2 \quad (65)$$

$$R_{\perp}(i) = \left[ \frac{\cos i - \sqrt{\frac{E_{0\parallel}}{E_{0\perp}} - \sin^2 i}}{\cos i + \sqrt{\frac{E_{0\parallel}}{E_{0\perp}} - \sin^2 i}} \right]^2 \quad (66)$$

Here again,  $R$  is invariant to nn. interchanges,  
and  $T$  is not.

At normal incidence,  $i=0$ , and eq. (64) gives

$$R_{\perp} = \frac{1 - \sqrt{\frac{\epsilon_m}{\epsilon_m + \mu}}}{1 + \sqrt{\frac{\epsilon_m}{\epsilon_m + \mu}}}^2$$

$$\approx \left[ \frac{\sqrt{\epsilon_m} - \sqrt{\epsilon_m}}{\sqrt{\epsilon_m} + \sqrt{\epsilon_m}} \right]^2 \quad n = \sqrt{\mu \epsilon}$$

If  $\mu = \mu_0$ ,  $n = \sqrt{\epsilon}$ , and

$$R = \left[ \frac{n - n_0}{n + n_0} \right]^2 \quad (67)$$

since  $R_{\parallel\perp} = R_{\perp}$  in this approximation. From eq. (56) for  $i=0$ ,

$$T_{\perp} = \left[ \frac{2}{1 + \sqrt{\frac{\epsilon_m}{\epsilon_m + \mu_0}}} \right]^2$$

$\text{For } \mu = \mu_0,$

$$T = \left[ \frac{2 n_0}{n + n_0} \right]^2 \quad (68)$$

the transmission coefficient. Again,  $T_{\perp} = T_{\parallel}$ .

Now consider eq. (65). The reflectivity  $R \perp$  to the plane of incidence will be zero when, for  $\mu = \mu_0$ ,

$$\frac{E/E_0}{\sqrt{\frac{\epsilon}{\epsilon_0} - \sin^2 i}} = 0.$$

$$\therefore \left(\frac{\epsilon}{\epsilon_0}\right)^2 = \frac{\epsilon}{\epsilon_0} - \sin^2 i;$$

$$\sin^2 i = \frac{\epsilon}{\epsilon_0} - \left(\frac{\epsilon}{\epsilon_0}\right)^2$$

$$\sin i = \sqrt{\frac{\epsilon}{\epsilon_0} - \left(\frac{\epsilon}{\epsilon_0}\right)^2}$$

$$\text{For } n = \sqrt{\epsilon}, \sin i = \sqrt{\frac{n^2}{n_0^2} - \left(\frac{n^2}{n_0^2}\right)^2} = \sqrt{\frac{n^2}{n_0^2} - \left(\frac{n^2}{n_0^2}\right)} \sqrt{\frac{n}{n_0} + \left(\frac{n}{n_0}\right)}$$

Since  $n > n_0$ ,  $\sin i$  has become imaginary. We discuss its significance presently. Now consider the reflectivity  $R \perp$  to the plane of incidence.  $R \perp = 0$  when, for  $\mu = \mu_0$ ,

$$\cos i - \sqrt{\frac{\epsilon}{\epsilon_0} - \sin^2 i} = 0$$

$$\cos^2 i = \frac{\epsilon}{\epsilon_0} - \sin^2 i$$

$$\frac{\epsilon}{\epsilon_0} = 1,$$

or when there is no interface at all. This is reasonable but nothing to write home about.

Let's now consider expression for the  $\vec{E}$ -vector, eq. (61) to (64) which we wrote in terms of  $i+r$  assuming  $\mu \neq \mu_0$ . From eq. (63), for H component,

$$E'/E_0 = \tan(i-r)/(\tan(i+r)), \quad (69)$$

we see that the reflectivity is zero when  $\tan(i-r) = 0$ , i.e., when  $i=r$ . From Snell's law this gives  $n_0 \sin i = n \sin r = n \sin i$ , or  $n = n_0$ , the same trivial result just obtained.

$E \perp$  to the plane of incidence again reproduces the result through the  $\sin(i-r)$  numerator. Now another way is to demand that, instead of  $\tan(i-r) = 0$  in eq. (69),  $\tan(i+r) = \infty$ .  $\therefore \cos(i+r) = 0$ .  $\therefore i+r = \pi/2$ .

If we cannot resolve the planet do the preferential reflection of  $E_s$  near the center and of  $E_t$  near the limb just cancel or are we left with a net polarization?

If we have a variation of reflectivity, ~~along~~ over the disk we expect a net polarization on reflection. In emission a variation of  $E$  or  $T$  or both over the disk will leave us with a net polarization to the emitted radiation.

If  $i = \frac{\pi}{2} - r$ , and  $n \sin i = n \sin r$ ,

$$n \sin i = n \sin\left(\frac{\pi}{2} - i\right) = n \cos i.$$

$$\therefore \tan i = \frac{n}{n_0}$$

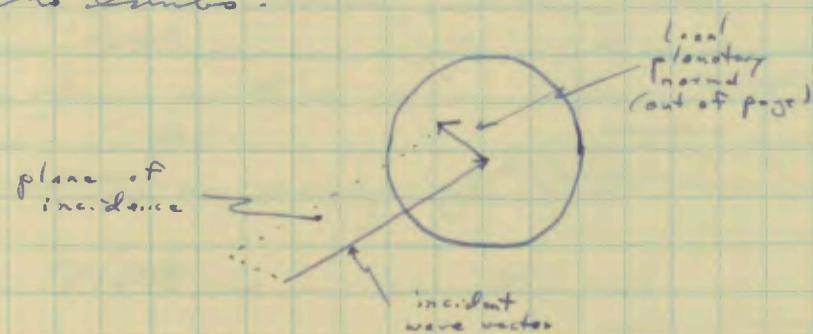
$$\therefore i_B = \arctan\left(\frac{n}{n_0}\right)$$

Brewster's angle  
For  $\vec{E} \parallel$  plane of incidence (70)

Thus the reflectivity is zero when the angle of incidence equals Brewster's angle  $i_B$ .

When  $\vec{E}$  is  $\perp$  to the plane of incidence, eq. (61) gives no non-trivial solution ... if a wave train is incident on an interface at  $\angle i = i_B$ , and the incident wave has mixed polarization, the reflected wave will be completely plane polarized with the polarization vector  $E_1$  perpendicular to the plane of incidence. This is a way of producing plane  $\parallel$  light.

Even at other angles of incidence, there is a tendency for the reflected wave to be predominantly polarized  $\perp$  to the plane of incidence. As a result, we expect different degrees of polarization from light reflected from the center of a planet, and from the limbs:

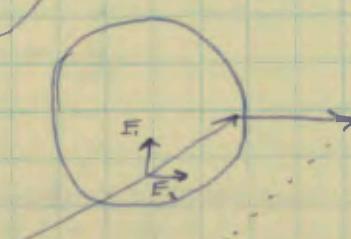


or, better,



$\therefore$  at center of disk  $E_R$  is preferentially reflected.

at limb,



$\therefore$  at limb,  $E_R$  is preferentially reflected.

This is the form used by Heiles & Drisko.

For  $n = \sqrt{3.67} = 1.92$ , as the radar data implies for  $\vartheta$ , and with  $n_0 = 1$ ,

$$i_B = \arctan 1.92 = 62.5^\circ.$$

For  $n = \sqrt{1.3} = 1.1$ , as some obs. of the Comteyart,

$$i_B = \arctan 1.1 = 47.7^\circ.$$

For a smooth sphere, Kirchhoff's law gives emissivities

$$\left. \begin{aligned} C_{\parallel}(i) &= 1 - R_{\parallel}(i) \\ C_{\perp}(i) &= 1 - R_{\perp}(i) \end{aligned} \right\} (71)$$

Thus especially near the Brewster  $\vartheta$ , but elsewhere as well,  $C_{\parallel} > C_{\perp}$ . . . from the pictures at the bottom of p. 73, we see that since the  $\vec{E}_i$  vector is primarily reflected at the limb the emission at the limb will be primarily in the direction of  $\vec{E}_i$  [i.e., along the planetary diameter]. A radiotelescope polarized as  $\vec{E}_\perp$ , should see preferential emission near the Brewster  $\vartheta$ , towards the limb. Polarized as  $\vec{E}_\parallel$ , it should see no emission deviating from the value of  $C_{\parallel}$ . This kind of argument suggests that with the accuracy of Kirchhoff's law, the sun information can be gained from passives as from actives. E.M. observations except where — as with a range gate — the time of the return is significant.

Now we can rewrite (70):

$$\frac{\sin i_B}{\cos i_B} = \frac{n}{n_0} = \sqrt{\frac{\epsilon}{\epsilon_0}} = \frac{\sqrt{1 - \cos^2 i_B}}{\cos i_B}$$

$$\therefore \frac{\epsilon}{\epsilon_0} = \frac{1 - \cos^2 i_B}{\cos^2 i_B} = \frac{1}{\cos^2 i_B} - 1$$

$$\therefore \frac{1}{\cos^2 i_B} = 1 + \frac{\epsilon}{\epsilon_0}$$

$$\therefore \cos i_B = \frac{1}{\sqrt{1 + \frac{\epsilon}{\epsilon_0}}} \quad (72).$$

$$n = \sqrt{\epsilon}$$

$$R = \left( \frac{n-1}{n+1} \right)^2$$

$$\epsilon = 1 - R$$

$\frac{\epsilon}{[0]}$	$n$ [0]	$R$ [0]	$\epsilon$
1	1	0	1
0.21	1.10	0.0083	0.998
0.77	1.33	0.0202	0.98
0.67	1.92	0.100	0.90
9	3	0.250	0.75
25	5	0.36	0.64
81	9	0.80	0.20
$\infty$	$\infty$	1.00	0.00

Or alternatively,  $T_{\infty}$  is the temp. of the source subject with assumption it is a black body, while  $T$  is its true temperature.

Note  $n \rightarrow \infty \Rightarrow$  perfect reflector; no emission.

Let us assume that the power reflected from a smooth planet is very close to that which would be reflected from the center of the disk. This is certainly true approximately, as the first Fresnel zone is small.

∴ from eqs. (65) & (66), for  $\mu = \mu_0$ ,  $\epsilon_0 = 1$

$$R_{\parallel} = \left( \frac{\epsilon - \sqrt{\epsilon}}{\epsilon + \sqrt{\epsilon}} \right)^2 = \left( \frac{\sqrt{\epsilon} - 1}{\sqrt{\epsilon} + 1} \right)^2$$

$$R_{\perp} = \left( \frac{1 - \sqrt{\epsilon}}{1 + \sqrt{\epsilon}} \right)^2$$

∴  $R_{\parallel} \approx R_{\perp}$  and the reflection is unpolarized. For a smooth surface, it follows that the emission is unpolarized, and that

$$\begin{aligned} \epsilon &= 1 - R = \frac{T_{BB}}{T_B} = 1 - \left( \frac{\sqrt{\epsilon} - 1}{\sqrt{\epsilon} + 1} \right)^2 \\ &= 1 - \left( \frac{n-1}{n+1} \right)^2 \end{aligned} \quad (73)$$

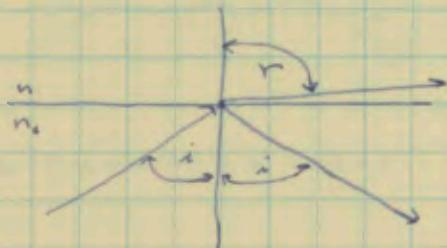
It is, as  $T/T_B$  suggests, the ratio of the brightness temperatures (by Wien's law  $\propto T^4$  & Rayleigh-Jeans  $\propto 1/\lambda^4$ ) actually observed, and that which would be observed from a black body,  $n = \sqrt{\epsilon} = 1$ . For the black body, of course,

$$R = \left( \frac{n-1}{n+1} \right)^2 \quad (74)$$

shows that the reflection would be zero if all the incident radiation being absorbed. As the dielectric constant increases, the total emitted power decreases. If the surface is perfectly smooth, if the  $i=0$  approximation is valid, the dielectric constant from the  $\perp$  center of a planetary disk could be determined from

the radar or loss reflectivity. Knowing the reflectivity from (74), the emissivity is obtained from Kirchhoff's law, & the true surface temperature can then be estimated from the brightness temp.

Now we consider another consequence of Fresnel's eqs., total internal reflection. The word internal signifies that the incident and reflected waves are in a medium of larger index of refraction than the refracted wave,  $n_0 > n$ . Snell's law,  $n_0 \sin i = n \sin r$  shows that, if  $n_0 > n$ , then  $r > i$ . Total internal reflection occurs when  $r = \pi/2$ :



Then,  $n_0 \sin i = n$ , or

$$\frac{i}{r} = \arcsin \frac{n}{n_0} \quad (75)$$

when a wave train is incident at  $i = i_s$ , the refracted wave is propagated along the interface, and no energy flows across this boundary.

What does  $i > i_s$  mean? For  $i > i_s$ ,  ~~$\sin r > 1$~~ ,  $\sin r > 1$ . E.g., for glass,  $n_0 = 1.33$ ,  $n = 1.0$ ,  $i_s = \arcsin 0.72 = 46^\circ$ . At an angle of incidence  $i = i_s = 46^\circ$ , no light is propagated out of the glass tube. This is the basis of the total internal reflection light pipe. For  $i > 46^\circ$ ,  $\sin r > 1.33 \sin i > 1$ . What does this mean?  $r$  must be a complex & with a purely imaginary cosine, thus:

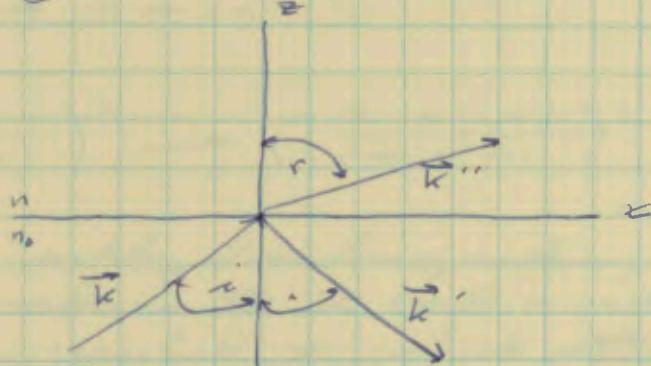
$$n_0 \sin i = n \sin r. \quad \frac{n}{n_0} = \frac{\sin i}{\sin r} = \sin i_s.$$

$$\therefore \sin^2 r = 1 - \cos^2 r = \frac{\sin^2 i}{\sin^2 i_s}.$$

$$\therefore \cos^2 r = 1 - \frac{\sin^2 i}{\sin^2 i_s}.$$

$$\therefore \cos r = i \sqrt{\left(\frac{\sin i}{\sin i_x}\right)^2 - 1} \quad (76)$$

Sin  $i > \sin i_x$ , by hypothesis,  $\cos r$  is purely imaginary. This situation becomes clear when we consider the propagation factor of the refracted wave:



$$e^{ik'' \cdot \vec{z}} = e^{ik''(x \cos r + z \cos i)}$$

$$\text{Now } \sin^2 r = 1 - \cos^2 r = 1 - \left[-\left(\frac{\sin i}{\sin i_x}\right)^2 + 1\right] = \left(\frac{\sin i}{\sin i_x}\right)^2$$

$$\begin{aligned} \therefore e^{ik'' \cdot \vec{z}} &= e^{ik''[\sin i / \sin i_x]^2 x} e^{ik''i[(\frac{\sin i}{\sin i_x})^2 - 1]^{1/2} z} \\ &= e^{-k''[(\sin i / \sin i_x)^2 - 1]^{1/2} z} e^{ik''(\sin i / \sin i_x) x} \end{aligned}$$

What does this say? For  $i > i_x$ , the sinusoidal factor applies only along the  $x$ -axis, and the wave is propagated along the interface for all  $i > i_x$ . In the  $z$ -direction, the wave is exponentially attenuated beyond the interface. Except for  $i = i_x$ , the attenuation occurs within a few wavelengths of the interface.

$$\frac{1}{e} \text{ attenuation when } z = \frac{1}{k''[(\sin i / \sin i_x)^2 - 1]^{1/2}}$$

$$k'' = n \frac{\omega}{c} = n \frac{2\pi c}{\lambda} = \frac{2\pi n}{\lambda}.$$

$$\therefore z = \frac{\lambda}{2\pi n [(\sin i / \sin i_x)^2 - 1]^{1/2}}$$

$\therefore z/\lambda$  is not a big no.

## Propagation of Electromagnetic Radiation in a Conducting Medium

Suppose the radiation is incident on a dielectric slab, at certain frequencies, is a total absorber. Then the index of refraction will be complex:

$$n \rightarrow n - i\kappa'$$

$$\therefore R = \left( \frac{n-i}{n+i} \right)^2 \rightarrow \left( \frac{n-i\kappa'-i}{n+i\kappa'+i} \right)^2$$

Thus, in a weakly absorbing medium,  $n' \ll n$ , and the reflectivity reduces to its real  $n$  value. Where a frequency is reached at which strong absorption exists,  $n' \gg n$ , and

$$R \approx \left( \frac{-i\kappa'}{i\kappa'+1} \right)^2 \rightarrow 1.$$

Thus, those frequencies which are strongly absorbed in transmission are strongly reflected in reflection.

The effect of complex  $n$  on the transmission coefficient, eq. (68), is

$$T \rightarrow \left[ \frac{2}{n - i\kappa' + i} \right]^2$$

For  $n' \gg n$ , denominator is large; transmission goes down, as we expect.

[Additional topics which could be introduced at this point: optical activity. v., e.g. Condon & Odishaw, 6-12, § 8]

## E. Propagation of Electromagnetic Radiation in a Conducting Medium

Since soil + rock have finite conductivities, we wish to modify the previous results accordingly. The medium is characterized by a conductivity  $\sigma$ , as well as the dielectric constant  $\epsilon$  and the permeability  $\mu$ . Maxwell's eqs, supplemented by Ohm's law

$$\vec{J} = \sigma \vec{E}$$

become

$$\begin{aligned} \vec{\nabla} \cdot \mu \vec{H} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\mu}{c} \frac{\partial \vec{H}}{\partial t} &= 0 \\ \vec{\nabla} \times \vec{H} - \frac{\epsilon}{c} \frac{\partial \vec{E}}{\partial t} &= \frac{4\pi}{c} \vec{J} = \frac{4\pi\sigma}{c} \vec{E} \end{aligned} \quad \left. \begin{array}{l} \vec{\nabla} \cdot \epsilon \vec{E} = 0 \\ \text{(no charge)} \end{array} \right\} (77)$$

To simplify, consider fields which vary in only one spatial variable,  $\xi$ . Decomposing the fields into longitudinal (" $\parallel \vec{k}$ ") and transverse ( $\perp \vec{k}$ ) parts, we have

$$\vec{E}(\xi, t) = \vec{E}_{\text{long}}(\xi, t) + \vec{E}_{\perp}(\xi, t)$$

$$\vec{H}(\xi, t) = \vec{H}_{\text{long}}(\xi, t) + \vec{H}_{\perp}(\xi, t)$$

The longitudinal components do not see the curl operator, and we have

$$\frac{\partial H_{\text{long}}}{\partial \xi} = 0$$

$$\frac{\partial E_{\text{long}}}{\partial \xi} = 0$$

$$\frac{\partial H_{\text{long}}}{\partial t} = 0$$

$$-\frac{\epsilon}{c} \frac{\partial E_{\text{long}}}{\partial t} = \frac{4\pi\sigma}{c} E_{\text{long}}$$

} (78)

The last eq,

$$\frac{\partial E_{\text{long}}}{\partial t} + \frac{4\pi\sigma}{\epsilon} E_{\text{long}} = 0$$

has the solution

$$E_{\text{long}}(\xi, t) = E_0 e^{-4\pi\xi t/\epsilon} \quad (79)$$

Thus, in the absence of an applied current density  $\vec{J}_0(t)$ , no static longitudinal fields can exist in a conducting medium. Since good conductors, such as copper have very high  $\sigma \sim 10^{17} \text{ sec}^{-1}$ , any initial longitudinal electrical field is readily damped out.

Now for the transverse fields, varying as

$$e^{i\vec{k} \cdot \vec{x} - i\omega t},$$

e.g. (78), as on page 43+44 gives, from the 3rd eqn.,

$$\vec{H} = \frac{c}{\mu\omega} (\vec{k} \times \vec{E})$$

$$i(\vec{k} \times \vec{H}) + \frac{\epsilon}{c} i\omega \vec{E} = \frac{4\pi\sigma}{c} \vec{E} \quad k = \omega/c.$$

... substituting for  $\vec{H}$ ,

$$i(\vec{k} \times \vec{k} \times \vec{E}) \frac{c}{\mu\omega} + i\frac{\epsilon}{c}\omega \vec{E} = \frac{4\pi\sigma}{c} \vec{E}$$

$$\text{Now } \vec{k} \times \vec{k} \times \vec{E} = k^2 \vec{E}$$

$$\therefore \left[ ik^2 \frac{c}{\mu\omega} + i\frac{\epsilon}{c}\omega - \frac{4\pi\sigma}{c} \right] \vec{E} = 0. \quad \frac{\mu\omega}{ic}$$

$$\therefore k^2 = \frac{\mu\omega}{ic} \frac{4\pi\sigma}{c} - \frac{4\pi\sigma}{ic} \cdot \frac{\epsilon}{c}\omega$$

$$= \mu\epsilon \frac{\omega^2}{c^2} \left( 1 + i \frac{4\pi\sigma}{\omega\epsilon} \right)$$

from  $\frac{1}{i} = -i$

The first term comes from the displacement current. The second from the conductor current. The sum of which for  $H$  is for  $\vec{E}$ .

When  $\sigma = 0$ , we require  $k = \sqrt{\mu\epsilon} \frac{\omega}{c}$ . This determines the sign in thus the square root of  $k$ .

$$\therefore k = \left[ \mu\epsilon \frac{\omega^2}{c^2} \left( 1 + i \frac{4\pi\sigma}{\omega\epsilon} \right) \right]^{1/2}$$

It is clear that the propagation vector is complex.

$$k = \alpha + i\beta$$

Now the square root of a complex quantity can be written

$$\sqrt{x+iy} = \pm \left[ \sqrt{\frac{x+z}{2}} + i\sqrt{\frac{r-z}{2}} \right]$$

where  $y > 0$ , and  $r = \sqrt{x^2+y^2}$ ; because, evidently,

$$\begin{aligned} \left[ \sqrt{\frac{x+z}{2}} + i\sqrt{\frac{r-z}{2}} \right]^2 &= \frac{x+z}{2} + \frac{z}{2}i\sqrt{(x+z)(r-z)} - \frac{r-z}{2} \\ &= \frac{x}{2} + \frac{z}{2} - i\sqrt{r^2-x^2} - \cancel{\frac{x}{2} + \frac{z}{2}} = x+iy \end{aligned}$$

$$\therefore k = \sqrt{\mu\epsilon} \frac{\omega}{c} \left[ \sqrt{\frac{1 + \left[ 1 + \left( \frac{4\pi\sigma}{\omega\epsilon} \right)^2 \right]^{\frac{1}{2}}}{2}} + i\sqrt{\frac{-1 + \left[ 1 + \left( \frac{4\pi\sigma}{\omega\epsilon} \right)^2 \right]^{\frac{1}{2}}}{2}} \right] \quad (80)$$

For a poor conductor,  $\frac{4\pi\sigma}{\omega\epsilon} \ll 1$ .

$$\therefore k = \sqrt{\mu\epsilon} \frac{\omega}{c} \left[ \sqrt{\frac{1 + \left[ 1 + \left( \frac{4\pi\sigma}{\omega\epsilon} \right)^2 \right]^{\frac{1}{2}}}{2}} + i\sqrt{\frac{-1 + \left[ 1 + \left( \frac{4\pi\sigma}{\omega\epsilon} \right)^2 \right]^{\frac{1}{2}}}{2}} \right]$$

$$= \sqrt{\mu\epsilon} \frac{\omega}{c} \frac{1}{\sqrt{2}} \left[ \sqrt{1 + \frac{8\pi^2\sigma^2}{\omega^2\epsilon^2}} + i\sqrt{\frac{8\pi^2\sigma^2}{\omega^2\epsilon^2}} \right]$$

$$= \sqrt{\mu\epsilon} \frac{\omega}{c} \left[ 1 + \frac{2\pi^2\sigma^2}{\omega^2\epsilon^2} + i \frac{2\pi^2\sigma^2}{\omega\epsilon} \right]$$

$$\approx \sqrt{\mu\epsilon} \frac{\omega}{c} \left[ 1 + i \frac{2\pi^2\sigma^2}{\omega\epsilon} \right]$$

$$k = \sqrt{\mu\epsilon} \frac{\omega}{c} \left[ 1 + i \frac{2\pi\sigma}{\omega\epsilon} \right] = \sqrt{\mu\epsilon} \frac{\omega}{c} + i \frac{2\pi}{c} \sqrt{\frac{\mu}{\epsilon}} \sigma \quad (81)$$

$$\omega = 2\pi\nu = 2\pi c/\lambda$$

$$\begin{aligned}\therefore k_x &= \sqrt{\mu\epsilon} \frac{2\pi}{\lambda} + i \frac{2\pi}{c} \sqrt{\frac{\mu}{\epsilon}} \sigma \\ &= \sqrt{\mu\epsilon} \frac{2\pi\nu}{c} + i \frac{2\pi}{c} \sqrt{\frac{\mu}{\epsilon}} \sigma \quad \text{For } \mu = 1 \\ k_v &= \frac{2\pi}{c} \left[ \sqrt{\epsilon} \nu + i \frac{\sigma}{\sqrt{\epsilon}} \right]\end{aligned}$$

~~$$|k_v|^2 = \frac{4\pi^2}{c^2} \left[ \epsilon \nu^2 + \frac{\sigma^2}{\epsilon} + 2\sigma\nu \right]$$~~

$$|k_v| = \frac{2\pi}{c} \left[ \epsilon \nu^2 + \frac{\sigma^2}{\epsilon} + 2\sigma\nu \right]^{1/2},$$

which reduces properly to  $\frac{2\pi\nu\sqrt{\epsilon}}{c}$  as  $\sigma \rightarrow 0$ .

For rock,  $\sigma = 3 \times 10^7 \text{ sec}^{-1}$ . For microwaves,  $\nu = \frac{3 \times 10^{10}}{3} = 10^{10} \text{ sec}^{-1}$ .  
 $\therefore \nu \gg \sigma$ , so  $k_v = \frac{2\pi\nu\sqrt{\epsilon}}{c}$ , one would think.

$$\text{Now } k = \sqrt{\mu\epsilon} \frac{\omega}{c} + i \frac{2\pi}{c} \sqrt{\mu\epsilon} \sigma, \text{ poor conductor} \quad (81)$$

Show that the conductivity enters only in the imaginary part. In this small  $\sigma$  limit,  $\text{Re } k \gg \text{Im } k$ . The attenuation of the wave ( $\text{Im } k$ ) is frequency-independent provided  $\sigma \neq \sigma(\omega)$ .

Actually, there is a frequency dependence of the lateral conductivity, especially at radio frequencies.

For a good conductor,  $\frac{2\pi\sigma}{\omega\epsilon} > 1$ , and

$$k = \sqrt{\mu\epsilon} \frac{\omega}{c} \left[ \sqrt{\frac{1}{2}} \sqrt{\left( \frac{2\pi\sigma}{\omega\epsilon} \right)^2 - \frac{1}{2}} + \sqrt{\frac{1}{2}} \sqrt{\frac{2\pi\sigma}{\omega\epsilon}} \right]$$

$$= (1+i) \sqrt{\mu\epsilon} \frac{\omega}{c} \sqrt{\frac{2\pi\sigma}{\omega\epsilon}} = (1+i) \sqrt{\mu} \sqrt{\omega} \sqrt{2\pi\sigma} \frac{1}{c}$$

$$\therefore k \approx (1+i) \frac{\sqrt{2\pi\omega\mu\sigma}}{c} \quad \text{good conductor} \quad (82)$$

Thus, for a good conductor, the propagation or wave vector is  $\propto \sqrt{\omega}$ , and is frequency-dependent, even if  $\sigma \neq \sigma(\omega)$ .

Now for waves propagating as  $e^{i\vec{k} \cdot \vec{x} - i\omega t}$ , the fields can be written as

$$\vec{E} = \vec{E}_0 e^{-\beta \vec{k} \cdot \vec{x}} e^{i\vec{k} \cdot \vec{x} - i\omega t}$$

Thus  $\beta$  gives a damping with distance, + a phase lag. In a good conductor, the  $\beta$ -term has magnitude

$$e^{-\frac{\sqrt{2\pi\omega\mu\sigma}}{c} x}$$

Thus,

$$\delta \approx \frac{c}{\sqrt{2\pi\omega\mu\sigma}}, \text{ good conductor} \quad (83)$$

$\therefore$  the skin depth. In high frequency waves

the current flows only on the surface of the conductor.  
 E.g. for Cu,  $\omega = 60 \text{ cps} \rightarrow \delta \approx 0.85 \text{ cm}$ ;  $\omega = 100 \text{ Mc/sec}$   
 $\rightarrow \delta \approx 7.1 \times 10^{-3} \text{ m}$ .

Now for a poor conductor, the expression is not the same. The  $\beta$  term has magnitude

$$e^{-\frac{2\pi}{c}\sqrt{\frac{\mu}{\epsilon}}\delta x}$$

Thus,

$$\boxed{\delta = \frac{c}{2\pi\sigma} \sqrt{\frac{\epsilon}{\mu}}, \text{ poor conductor}} \quad (84)$$

A poor conductor is here defined as  $\frac{4\pi\sigma}{\omega\epsilon} \ll 1$ . It can be shown that the phase lag which it lags  $\vec{E}$  is

$$\phi_0 = \operatorname{arctan} \frac{\beta}{\omega} = \frac{1}{2} \operatorname{arctan} \left( \frac{4\pi\sigma}{\omega\epsilon} \right)$$

$\therefore \phi_0$  is very small for poor conductors. In addition there is a progressive phase lag with depth in conductors, just as in the heat conduction problem. The time dependent part of the soln. for the flux is of the form

Factor  
2 from  
 $E^2$

$$\cos(\omega t - \alpha z)$$

which, for poor conductors reads

$$\begin{aligned} \cos(\omega t - \sqrt{\mu\epsilon} \frac{\omega}{c} z) &= \cos(\omega t - \frac{n}{c} \omega z) \\ &= \cos \omega \left( t - \frac{n}{c} z \right) \end{aligned} \quad (85)$$

There should also be an arbitrary initial phase  $\delta$ .

$$\text{cm} \cdot \frac{\text{cm/sec}}{\sigma} \quad \sigma = \text{sec}^{-1}$$

The average skin depth is  $\frac{1}{2}$  the full skin depth.  
 $\therefore$  for a paraboloid,

$$\delta_E = \frac{1}{2} \delta = \frac{c}{4\pi\sigma} \sqrt{\frac{\epsilon'}{\mu}} \quad (86)$$

A typical value of geometrical factor is

$$\sigma = 3 \times 10^7 \text{ sec}^{-1}$$

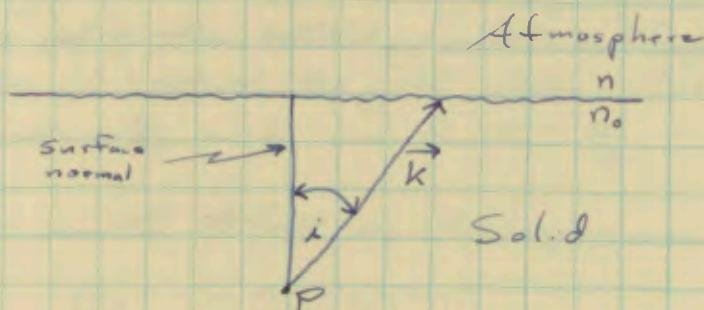
$$\therefore \delta_E \approx \frac{3 \times 10^{10}}{4\pi \times 3 \times 10^7} = 0.08 \times 10^3 = 80 \text{ cm} \quad (87)$$

$\therefore$  the paraboloid skin depth is of the same order as the given wavelength.

Is this not the effective depth of emission? And if so, doesn't it follow that all microwave wavelengths come from the same depth except for  $\sigma = \sigma(\omega)$ ? No. E.m. radiation cannot come from deeper than a few skin depths. But short wavelengths may come from shallower depths. What eq. (87) says is that long wavelength emission may come from a region of depth  $\delta \ll \lambda$ .

Query: what is  $n$  for  $P \approx 50$  atm?

## F. Equation of Radiative Transfer in a Solid



Consider an electromagnetic wave train incident on a solid-gas interface from below. Since, even for  $\text{F}$  with its dense atmosphere, we expect  $n_0 > n$ , we have the phenomena of total internal reflection (p. 79 et seq.). Only the thermal radiation is a cone centered about the surface normal, and with a half-angle

$$i_s = \arcsin \frac{n}{n_0}$$

will escape from the solid. One might therefore tend to believe that the brightness temperature will, from this cause, be very much less than the true temperature. Let us explore this possibility for a moment. Let  $T$  be the "true" (i.e., thermometric) temperature, +  $T_B$  the brightness temperature of a small plane area.

$$T_B = \epsilon T = (1 - R)T = \left[1 - \left(\frac{n_0 - 1}{n_0 + 1}\right)^2\right]T \quad (88)$$

for  $n = 1$  + Kirchhoff's law. Now the radiation escaping from pt. P is  $\sigma$  the area of the base of the right circular cone with P at the apex +  $i$  the half-angle. Thus the emission, and — by the Rayleigh-Jeans law for microwave frequencies — the brightness temperature is a  $\sin^2 i$ .

$$\therefore \frac{T_B}{T} = \frac{\sin^2 i}{\sin^2 90^\circ} = \left(\frac{n}{n_0}\right)^2 = \frac{1}{n_0^2} \quad \text{for } n = 1.$$

$$\therefore T_B = \frac{1}{n_0^2} T. \quad (89)$$

Both (88) + (89) have the properties  $n_0 \rightarrow \infty, T_B \rightarrow 0$ ;  $n_0 \rightarrow 1, T_B \rightarrow T$ . Otherwise do they agree?  $n_0 = 2 \Rightarrow T_B/T = [1 - (\frac{1}{3})^2] = 8/9$  from (88) and  $\frac{1}{4}$  from (89). — the problem of total internal reflection in thermal emission from a solid is not solved this

Query: Is it clear that the effect of the emissivity, eq. (88) is not such a "geometrical" effect?

Query: So far normal incidence were cool. What about oblique incidence, eqs. (65)+(66)? Done. See p. 66.

$$\text{Numerator of (65) is } \left[ \frac{\epsilon_{\mu_0}}{\epsilon_{\mu} \mu} - \sqrt{\frac{\epsilon_{\mu_0}}{\epsilon_{\mu} \mu}} - \sin i \right]^2. \text{ But } \sqrt{\epsilon_{\mu_0}} \sin i = \sqrt{\epsilon_{\mu}} \sin$$

$$\therefore \left[ \right]^2 = \left[ \frac{\epsilon_{\mu_0}}{\epsilon_{\mu} \mu} - \sqrt{\frac{\epsilon_{\mu_0}}{\epsilon_{\mu} \mu}} - \frac{\sqrt{\epsilon_{\mu}} \sin i}{\sqrt{\epsilon_{\mu_0}}} \right]^2. \text{ Now divide by } \sqrt{\epsilon_{\mu_0}}, \text{ i.e., multiply by } \left[ \frac{\epsilon_{\mu_0}}{\epsilon_{\mu}} \right]^2$$

$$\therefore \left[ \right]^2 = \left[ \frac{\epsilon_{\mu_0}}{\epsilon_{\mu} \mu} \sqrt{\frac{\epsilon_{\mu_0}}{\epsilon_{\mu}}} - \sqrt{\frac{\epsilon_{\mu}}{\mu^2}} - \frac{\sqrt{\epsilon_{\mu}} \sin i}{\sqrt{\epsilon_{\mu_0}}} \right]^2 \times \left[ \frac{\epsilon_{\mu_0}}{\epsilon_{\mu}} \right]^2. \text{ For } \mu = \mu_0,$$

$$\left[ \right]^2 = \left[ \sqrt{\frac{\epsilon_{\mu_0}}{\epsilon_{\mu}}} - \cos i \right]^2 \times (\sqrt{\frac{\epsilon_{\mu_0}}{\epsilon_{\mu}}})^2$$

simply. The problem is to counteract the effect of the  $\frac{1}{n_0^2}$  term in eq. (89). This is achieved by setting the same factor  $\propto n_0^2$ . I.e., if for another reason,  $T_0$  is also  $\propto n_0^2$ , the these "geometrical" effects are cancelled and we can attribute all differences  $T_0$  and  $T$  to the emissivity and conductivity.

Now note that such expression as

$$R = \left( \frac{n - n_0}{n + n_0} \right)^2$$

for the reflectivity are symmetric in  $n$  and  $n_0$ . I.e., the same reflectivity is experienced by a wave from entering the interface from below as from above. i.e., e.g., for a black body, emitting at temp.  $T$ , the radiation escaping to space from below the surface will be

$$(1 - R) \sigma T^4$$

This is just the emissivity-corrected emission for a grey body.  $\therefore C = (1 - R)$  arises, not at the moment of emission, but upon traversing the interface.

Suppose we had a reservoir at const. temperature  $T$  surrounding the surfaces at the one temp. at the surface (i.e., thermal equilibrium is established). If nothing counteracted total internal reflection, the temp. emitted to space would be  $\sin^{-1} \frac{n_0}{n} T$  less than the temp. of the emitter. Does this contradict with Stefan's law? But we will see later that inside the cavity emission is up by a factor  $n^2$ , just balancing the total internal reflection effect.

Our insight on the nature of emissivity obtained above shows us that the energy leaving from the heat bath is diminished by just the one factor as the energy leaving the solid.

100

Solution

Total internal reflection problem: See p. 103 et seq.

Taking now as constants, the angle of incidence,  $i$ , and the depth,  $z$ , the eq. of radiative transfer becomes

$$\cos i \frac{\partial I_v(i, z)}{\partial z} = -k_v I_v(i, z) + j_v \quad (90)$$

where  $k_v$  &  $j_v$  are, respectively, the absorption & emission coefficient per gram. The formal solution of (90) at  $z=0$  is

$$I_v(i, 0) = \int_0^\infty \frac{j_v}{k_v} e^{-k_v(\sec i) z} k_v(\sec i) dz \quad (91)$$

where  $S_v = j_v/k_v$  and we have assumed  $k_v \neq k_v(z)$ . In thermodynamic equilibrium, Kirchhoff's law with the ~~sin + i~~ condition gives

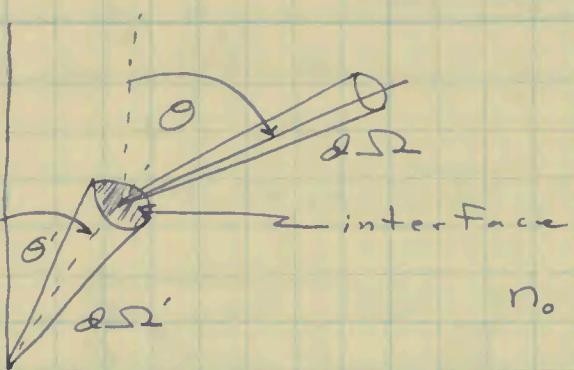
$$\frac{j_v}{k_v} = n_o^2 B_v(T) \quad (92)$$

where  $B_v(T)$  is the Planck distribution. For radio frequencies and  $\lambda$  surface temperatures, the Rayleigh-Jeans approximation holds.

$$B_v(T) = 2kT/\lambda^3 \quad (93)$$

Note that  $B_v d\omega = B_\lambda d\lambda$ . Substituting (92) & (93) in (91), we obtain

$$I_v(i, 0) = \frac{2k}{\lambda^3} n_o^2 \int_0^\infty T(z) e^{-k_v(\sec i) z} k_v(\sec i) dz \quad (94)$$



Flux seen here:  $I_n(\theta) \cos \theta d\Omega$

No. Flux leaving here:  
 $[1 - R(\theta)] I_o(\theta') \cos \theta' d\Omega'$

$\theta + \theta'$  related by Snell's law

Now consider a unit element of surface. Let  $R(i)$  be the power reflectivity, obtained from a  $\sqrt{S}$  of the sum of squares of the orthogonal polychromatic components, e.g. (65) + (66). Let  $I_v(i)$  be the specific intensity of radiation ~~emitted~~ from the interface ~~from above~~ (parallel to ~~the~~ plane), and  $I_v'(i')$  be the specific intensity ~~incident~~ from below.

A photometer, sitting in space above the interface and oriented "to it", sees a flux incident from ~~above~~ of

$$I_v(i) \cos i \, d\Omega$$

The emergent flux from the solid angle

$$[1 - R(i')] I_v'(i') \cos i' \, d\Omega'$$

(A photometer imbedded in the surface sees the same intensities, but with above+below reversed, & prime interchanged.)

In thermodynamic equilibrium,

$$[1 - R(i')] I_v'(i') \cos i' \, d\Omega' = I_v(i) \cos i \, d\Omega_{(95)}$$

By Snell's law,

$$n \sin \theta i' = \sin i$$

Letting  $n_\infty = 1$ , and here setting  $n_0 = n$ . By definition,

$$d\Omega = \sin \theta \, d\theta \, d\varphi$$

$$d\Omega' = \sin \theta' \, d\theta' \, d\varphi'.$$

By symmetry,  $\theta = \theta'$ . These  $\theta$ 's are the same as on previous i's.

$$\text{For normal incidence, } I_{v(0)}/I_{v'(0)} = \frac{n^2 + 2n + 1 - n^2 + 2n - 1}{n^2(n+1)^2}$$

$$= \frac{4n}{n^2(n+1)^2} = \frac{4}{n(n+1)^2}.$$

$n \rightarrow 1$ ,  $I_{v(0)}/I_{v'(0)} \rightarrow 1$ .  $\checkmark$ .  $n \rightarrow \infty$ ,  $I_{v(0)}/I_{v'(0)} \rightarrow 0$ .  $\checkmark$ .

$$\underbrace{I_{v'(0')} [1 - R(\theta')]}_{\text{Radiation emitted by the solid surface, converted from intensity } c = 1 - R \text{ at the interface, } \cancel{\text{this is right}}.} = \cancel{n^2 I_{v(0)}}$$

Radiation emitted by the solid surface, converted from intensity  $c = 1 - R$  at the interface, ~~this is right~~.

Radiation emitted from the surface w. Kirchhoff law converts

Since  $\frac{n^2}{1 - R(\theta')} > 1$ ,  $I_{v'(\theta')} > I_{v(\theta)}$ ; i.e., the radiation incident on the interface from below is ~~more intense than the~~ radiation which makes it out.

The attenuation is due to two effects: total internal reflection beyond  $\theta_s$ , and non-zero reflection within the exit cone.

Substituting in (95), we have

$$I_v(\theta) = I_v'(\theta') \frac{[1 - R(\theta')] \sin \theta' \cos \theta' d\theta' d\phi'}{\sin \theta \cos \theta d\theta d\phi}$$

From Snell's law,

$$n \sin \theta' = \sin \theta$$

$$\cos \theta' d\theta' = \frac{1}{n} \cos \theta d\theta$$

$$\therefore I_v(\theta) = I_v'(\theta') \frac{[1 - R(\theta')] \frac{1}{n} \sin \theta \cos \theta d\theta}{\sin \theta \cos \theta}$$

$$\therefore I_v(\theta) = I_v'(\theta') \frac{[1 - R(\theta')]}{n^2} \quad (46)$$

and here is our desired  $n^2$  factor? Now eq. (94) can be written, using ' to indicate radiation incident from below,

$$I_v'(\theta;_0) = \frac{2k}{\lambda^2} n^2 \int_0^\infty T(z) e^{-k_v z \sec \theta'} k_v (\sec \theta') dz$$

Substituting from (96), we have

$$\frac{n^2}{[1 - R(\theta')]} I_v(\theta) = \frac{2k}{\lambda^2} n^2 \int_0^\infty$$

$$\therefore I_v(\theta, 0) = [1 - R(\theta')] \frac{2k}{\lambda^2} \int_0^\infty T(z) e^{-k_v \sec \theta' z} k_v \sec \theta' dz \quad (47)$$

Suppose we surround our emitting slab, at temp.  $T$ , with an iso heat reservoir also at temp.  $T_r$ , and wait for thermodynamic equilibrium. Then  $T = T_r$ . The specific intensity entering the slab from the reservoir is

$$I_{vr} = [1 - R(\theta)] B_v(T_r).$$

Because  $R(\theta) = R(\theta')$  the specific intensity leaving the slab must also be

$$I_v = [1 - R(\theta)] B_v(T_r).$$

From (99) we see that this is the case only when

$$S_v = \frac{j_v}{k_v} = n^2 B_v(T).$$

To the extent that  $R(\theta) = R(\theta')$ , eq. (97) can be written  
i.e., always

$$I_v(\theta, 0) = [1 - R(\theta)] \frac{2k}{\lambda^2} \int_0^\infty T(z) e^{-k_v \sec \theta' z} k_v \sec \theta' dz, \quad (98)$$

an expression for the specific intensity incident from below. When  $T(z) = T = \text{const}$ , (98) becomes

$$I_v(\theta, 0) = [1 - R(\theta)] \frac{2kT}{\lambda^2} \int_0^\infty e^{-az} dz$$

$$\int_0^\infty e^{-az} dz = [e^{-az}]_0^\infty = +1.$$

$$\therefore |I_v(\theta, 0)| = [1 - R(\theta)] \frac{2kT}{\lambda^2} \quad (99)$$

as it must be at zero concentration. (99) gives the incident radiation from above as the product of the emitted radiation from below and the emissivity of the emitter.

G. Relation of Brightness Temperature,  $T_B$   
to Surface Temperature,  $T_s$ :

From p. 17, we have the diurnal solar period on Earth as  $P_{\text{sol}} = 118^\circ$ . From p. 15, eq. (8), we have

$$T_a(t) = T' + T_0 e^{-kt} \cos(\omega t - kx - \varphi) \quad (99a)$$

for a driving temperature at the surface, due to insolation, of

$$T_s(t) = T' + T_0 \cos(\omega t - \varphi).$$

$$\text{where } \omega = 2\pi/P_{\text{sol}}.$$

A black-body at a temperature  $T_B$  no sun from direction  $\theta$  emits in the microwave region an intensity

$$I_\nu(\theta) = \frac{2k}{\lambda^2} T_B(\theta). \quad (100)$$

Alternatively, eq. (100) serves as the definition of the equivalent black-body temperature or brightness temperature of any real body. Setting the eqs. in (98), we have, for  $\varphi=0$ ,

$$\begin{aligned} \frac{2k}{\lambda^2} T_B(\theta) &= [1 - R(\theta)] \frac{2k}{\lambda^2} \int_0^\infty [T' + T_0 e^{-kt} \cos(\omega t - kx - \varphi)] \\ &\quad \cdot e^{-K_\nu \sec \theta' \cdot z} K_\nu \sec \theta' dz \end{aligned}$$

It is important here to distinguish between the absorption coefficient  $K_\nu$  and the wave number of the reported wave  $k = \sqrt{\frac{\omega}{c\varepsilon_0}}$ . Neither of these is the c.m. propagator with magnitude  $|k|$ .

$$\text{Setting } \cos(\omega t - kx) = \text{Re } e^{i(\omega t - kx)}$$

we have, neglecting the phase  $\varphi$  for the moment,

$$\delta \cos^2 \theta + \cos \theta - i \delta \cos \theta + i \sin \theta + i \sin \theta + \gamma \sin \theta \cos \theta /$$

$$\frac{\delta^2 \cos^2 \theta + \delta \cos \theta - i \delta^2 \cos^2 \theta - 3 \cos \theta + 1 - i \delta^2 + i \delta^2 \cos^2 \theta + \sqrt{\delta^2 + \delta^2 \cos^2 \theta}}{2 \delta \cos^2 \theta + 2 \delta \cos \theta + 1}$$

$$T_a(\theta) = [1 - R(\theta)] \int_0^\infty [T' + T_0 e^{-kz} e^{i(wt - kz)}] e^{-k_v \sec \theta' z} k_v \sec \theta' dz$$

which becomes two integrals. The first is  $[1 - R(\theta)] T'$ .  
The second gives

$$[1 - R(\theta)] T_0 k_v \sec \theta' \int_0^\infty e^{iwt} e^{-(k + ik + k_v \sec \theta')z} dz$$

$$= [1 - R(\theta)] T_0 k_v \sec \theta' \operatorname{Re} \frac{\cos wt + i \sin wt}{k + ik + k_v \sec \theta'}$$

$$= [1 - R(\theta)] T_0 \cancel{\sec \theta'} \operatorname{Re} \frac{\cos wt + i \sin wt}{\left(\frac{k}{k_v}\right) \cos \theta' + i \left(\frac{k}{k_v}\right) \cos \theta' + 1}$$

$$\text{Now } \operatorname{Re} \frac{\cos wt + i \sin wt}{\left(\frac{k}{k_v}\right) \cos \theta' + i \left(\frac{k}{k_v}\right) \cos \theta'}$$

$$= \operatorname{Re} \frac{\cos wt + i \sin wt}{\left(\frac{k}{k_v}\right) \cos \theta' + 1 + i \left(\frac{k}{k_v}\right) \cos \theta'} \frac{\left(\frac{k}{k_v}\right) \cos \theta' + 1 - i \left(\frac{k}{k_v}\right) \cos \theta'}{\left(\frac{k}{k_v}\right) \cos \theta' + 1 - i \left(\frac{k}{k_v}\right) \cos \theta'}$$

$$= \operatorname{Re} \frac{\cos wt + \left(\frac{k}{k_v}\right) \cos \theta' \cos wt + \left(\frac{k}{k_v}\right) \sin \theta' \sin wt + i \left[\sin wt - \left(\frac{k}{k_v}\right) \cos \theta' \cos wt - \left(\frac{k}{k_v}\right) \cos \theta' \sin wt\right]}{2\left(\frac{k}{k_v}\right)^2 \cos^2 \theta' + 2\left(\frac{k}{k_v}\right) \cos \theta' + \cancel{1} + \cancel{i}}$$

$$= \frac{\cos wt [1 + \gamma \cos \theta'] + \gamma \sin wt \cos \theta'}{2\gamma^2 \cos^2 \theta' + 2\gamma \cos \theta' + \cancel{1}} \quad \text{where } \gamma = k/k_v.$$

$$\text{Now } a \cos wt + b \sin wt = r \cos (wt - \gamma^\circ)$$

$$\text{where } r = \sqrt{a^2 + b^2}, \gamma^\circ = \arccos \frac{a}{r} = \arcsin \frac{b}{r}$$

$$\text{Here } a = 1 + \gamma \cos \theta' + b = \gamma \cos \theta'$$

$$\therefore r^2 = 1 + 2\gamma \cos \theta' + \gamma^2 \cos^2 \theta'$$

$$\therefore \text{the second } S \text{ is the } \frac{\cos (wt - \gamma^\circ)}{[1 + 2\gamma \cos \theta' + 2\gamma^2 \cos^2 \theta']^{1/2}}$$

where  $\gamma^\circ$  is given as above. It can be written as follows:

Or, reinserting the phase  $\varphi \equiv \alpha_0$ , we have

$$T_B(\theta) = [1 - R(\theta)] \left\{ T_0 + \frac{T_0 \cos(\omega t - \alpha_0 - \gamma^\circ)}{[1 + 2\delta \mu' + 2\delta^2 \mu'^2]^{1/2}} \right\}$$

where  $\mu' = \cos \theta$ . To prevent the tedious proliferation of temperatures let us introduce

$$T_m(z; \xi; \eta; \theta)$$

i.e., the  $m$ th Fourier component of the temp. at depth  $z$ , latitude  $\xi$ , longitude  $\eta$ , and time  $\theta$ . The time dependence will ordinarily be suppressed. Here we also suppress the  $\xi + \eta$  dependence. There is also a  $\lambda$ -dependence. we have for the  $z$ -integrated brightness temp.,

$$T_B(\theta) = [1 - R(\theta)] \left\{ T_0(\theta) + \frac{T_0(\theta) \cos(\omega t - \alpha_0 - \gamma^\circ)}{[1 + 2\delta \mu' + 2\delta^2 \mu'^2]^{1/2}} \right\}$$

$\delta = k/k_v$ , the wave no. of the thermal wave  $\kappa_{\text{max}}$  absorption coefficient.

$$\gamma^\circ = \arctan \frac{1 + \delta \mu'}{\delta \mu'} = \arctan [1 + (\delta \mu')^2].$$

$k = \sqrt{\frac{\omega}{2\pi c}}$ . For a poor conductor,  $k_v = \sqrt{\frac{2\pi}{c}} \frac{2\pi}{\epsilon} [\sqrt{\epsilon} \nu + i \frac{\sigma}{\epsilon}]$ .

$+ |k_v| = \frac{2\pi}{c} \left[ \epsilon \nu^2 + \frac{\sigma^2}{\epsilon} + 2\sigma \nu \right]^{1/2}$ . Here  $\nu$  is on frequency,  
 $\omega$  is <sup>planetary</sup> rotation circular frequency.

$$\cos \gamma = \frac{a}{r}, \sin \gamma = \frac{b}{r} \therefore \tan \gamma = \frac{a}{b}$$

$$\therefore \gamma = \arctan \frac{1 + \gamma \cos \theta'}{\gamma \cos \theta'} \quad (101)$$

Collecting terms, then, we find

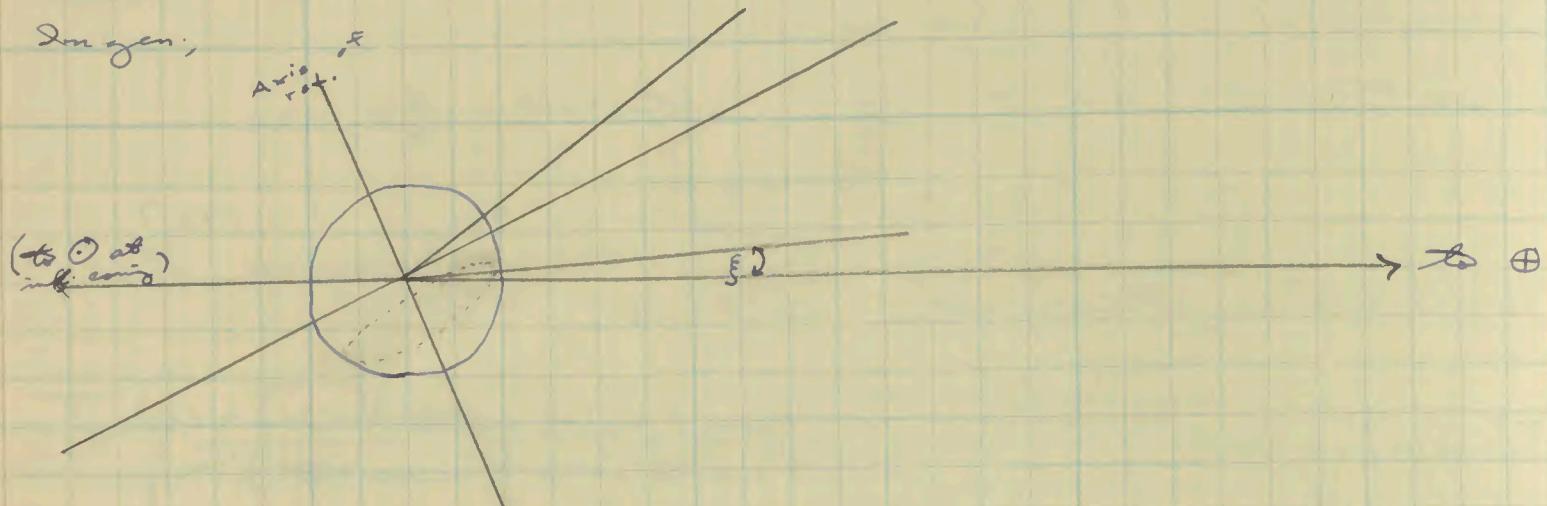
$$T_B(\theta) = [1 - R(\theta)] \left\{ T' + \frac{T_0 \cos(\omega t - \gamma)}{[1 + 2\gamma \cos \theta' + 2\gamma^2 \cos^2 \theta']^{1/2}} \right\} \quad (102)$$

Eq. (102) relates the brightness temp. of an object with a surface driving temp  $T' + T_0 \cos \omega t$ , observed at a  $\theta'$ ; instead of the driving frequency  $\omega$ , the index of refraction of the object (contained in  $R$ ), the monochromatic absorption coefficient  $k_\nu$  at the frequency  $\nu$  while  $T_0(\theta)$  is measured, and the wave number of the temperature wave,  $\kappa = \sqrt{\frac{\omega}{2\pi c}}$ . Alternatively,  $T_0(\theta)/[1 - R(\theta)]$  is the true temperature it would have in the emission depth. The  $\gamma$  term puts the observed temperature variation out of phase with the driving temperature variation. Eq. (101) gives us some info. on  $\delta, \gamma$ : on the surface component [By hypothesis  $\delta \neq \delta(z)$ ]. The factor  $[ ]^{1/2}$  is always  $> 1$ .  $\therefore$  the amplitude of the time varying term observed is always less than that of the driving  $T$ . Since the amplitude declines with depth, + we have integrated over  $z$ , this circumstance is entirely reasonable.

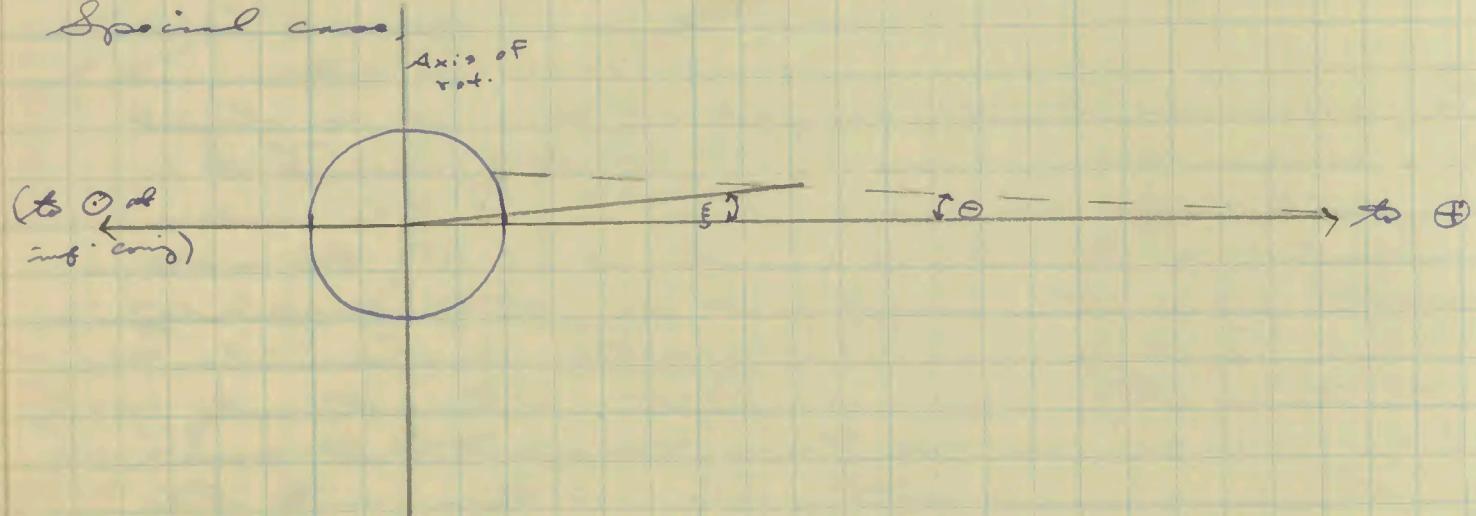
Now we want to establish a coordinate system to describe the position of a point on the surface of  $\mathbb{S}$ . We desire polar coordinates  $\xi$  and  $\eta$ , corresponding to the womb  $\theta$  and  $\phi$ . The polar axis, about which  $\eta$  is measured, is taken as the Cylindrical rotation axis. The surface normal to the center of the  $\mathbb{S}$  has coordinates  $\xi = 0, \eta = 0$ , or that  $180^\circ$  plane & corresponds to  $\eta = 0$  ( $\Rightarrow \xi = 0$   $\Rightarrow$  the condition that the orbits of  $\theta + \phi$  are coplanar). We take  $\theta = 0$  when plane  $\xi = 0^\circ$ . This causes the introduction of a plane  $\eta = 0$ .  $\Delta$  is the argument of the cosine in eq. (102). Alternatively, we could

114

In gen.



Special case,



Query: Is  $\alpha$  different from  $\theta$ ? Why is  $\alpha$  present in the argument of the cosine?

have retained the phase and  $\theta = \text{sgn. } (\varphi - \alpha)$ . The phase const.  $\delta$  allows for the diff. between local noon and the time of max. temp. Clearly,  $\delta = \delta(k)$ . The const. in the  $T$  division formula,  $T'$  and  $T_0$  are clearly functions of position on the disk of  $\varphi$ . In the case that  $\varphi$  has an axial incl. of  $0^\circ$ ,  $T'$  and  $T_0$  are clearly functions only of  $\xi$ . The  $\eta$  dependence of the  $T$  at a given pos. is given by  $w\theta$ . But for non-zero inclination const.  $\beta$  cuts a small  $\Omega$  on the sphere of  $\varphi \parallel$  to the rotation axis. The  $\eta$  variation can again be written in terms of a w $\theta$  width. In both cases,  $T' = T'(\xi)$  and  $T_0 = T_0(\xi)$ . More explicitly,

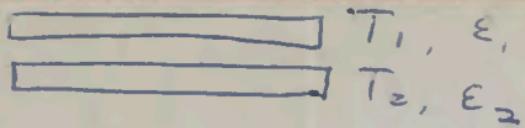
$$T' = f(\xi) T_{eq} \quad T_0 = f(\xi) T_{0q} \quad (103)$$

Here eq indicates equatorial values; i.e., when  $\xi = 0$ . Since that is when the inclination is minimum (for zero axial inclination),  $f(0) \leq 1$  and  $f(0) = 1$ . It is reasonable that  $T'$  and  $T_0$  have the same  $\xi$ -dependence. Presumably  $f(\xi)$  decreases monotonically as  $\xi$  goes from 0 to  $\pm \pi/2$ .

Because the angular diameter of  $\varphi$  is generally much smaller than the radiotelescope beam width, the observed brightness temperature of  $\varphi$  will be

$$T_{obs} = \int_{\Omega_\varphi} [1 - R(\theta)] f(\xi) \left\{ T_{eq}' + \frac{T_{0q} \cos(w\theta - \gamma - \alpha - \frac{\pi}{4})}{[1 + 2\sin^2 \theta]^{1/2}} \right\} \frac{d\Omega}{\Omega_\varphi} \quad (104)$$

$d\Omega$  is the solid angle subtended by a differential element of area on  $\varphi$  as seen from  $\oplus$ .



$$I = \epsilon_1 b(T_1)(1 - \epsilon_2) + \epsilon_2 b(T_2)$$

$$+ \epsilon_2 b(T_2)(1 - \epsilon_1)(1 - \epsilon_2)$$

Let  $T_1 = T_2 = T$

$$\therefore I = b(T) \{ \epsilon_2 [ -\epsilon_1 + 1 + 1$$

$$- \epsilon_2 - \epsilon_1 + \epsilon_1 \epsilon_2 ] + \epsilon_1 \}$$

$$= b(T) \{ \epsilon_2 [ 2(1 - \epsilon_1)$$

$$+ \epsilon_2 (\epsilon_1 - 1) ] + \epsilon_1 \}$$

$$= b(T) \{ \epsilon_2 (1 - \epsilon_1)(2 - \epsilon_2) + \epsilon_1 \}$$

emission  
only

$$\text{Let } \varepsilon_1 = 1 - s$$

$$\begin{aligned}\therefore I &= b(T) \left\{ \varepsilon_2 s (2 - \varepsilon_2) + 1 - s \right\} \\ &= b(T) \left\{ s [\varepsilon_2 (2 - \varepsilon_2) - 1] + 1 \right\}\end{aligned}$$

$$\varepsilon_2 = \frac{1}{2} \Rightarrow I = b(T) \left\{ 1 - \frac{s}{4} \right\}$$

$$s = 0.10 \rightarrow \{ 3 = 1 - 0.025$$

$$s = 0.05 \rightarrow \{ 3 = 1 - 0.012$$

$$\varepsilon_2 = \frac{2}{3} \Rightarrow I = b(T) \left\{ 1 - \frac{s}{9} \right\}$$

$$\varepsilon_2 = 1 \Rightarrow I = b(T)$$

$\therefore$  where atmosphere is not transparent  
need to discriminate a few %.

For  $\varepsilon_2 = \frac{1}{2}$ , need ~~4 x better~~  
accuracy.

$$\frac{I}{I_0} = \frac{1 - \frac{s}{4}}{1 - \frac{s}{2}}$$

